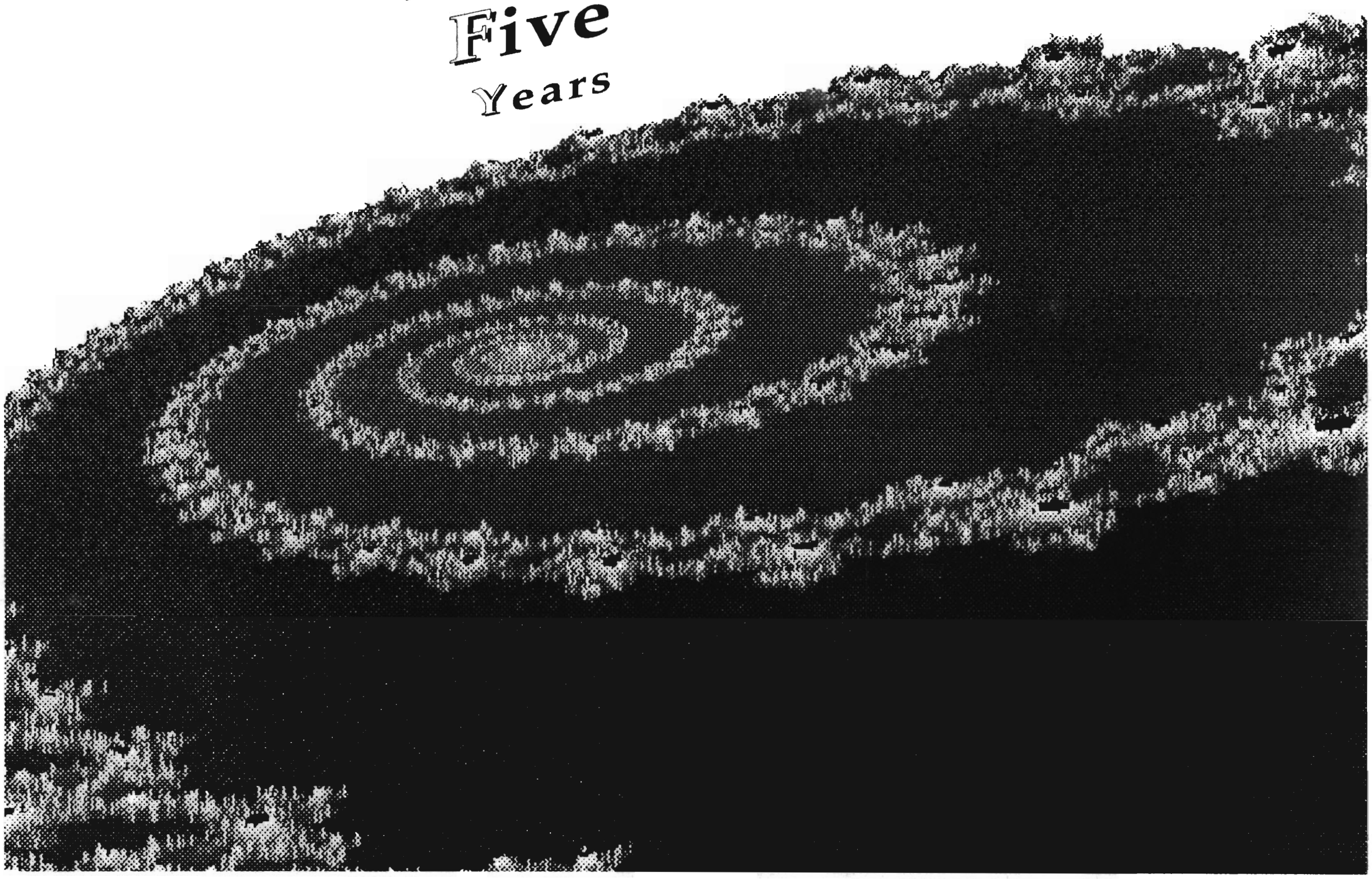
The First Five Years



A compilation of team tests from

The Mandelbrot Competition, 1990-1995

Sam Vandervelde

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This book was produced as camera ready copy using the LaTeX typesetting system.

To mom and dad
for all their
encouragement

To the student

It was six summers ago that I received a phone call from Richard Rusczyk inviting me to take part in an exciting venture. He wanted to create a mathematics competition for high school students and wondered if I would like to join Sandor Lehoczky and himself in a partnership to produce this contest. So far they had named the partnership: Greater Testing Concepts. I thought it was a crazy idea.

Within a week I called Richard back.

"I've been playing around with this neat iterative process lately which produces the most fantastic picture, it's called the Mandelbrot set, named after Benoit Mandelbrot who discovered it. Maybe we could call the contest the Mandelbrot Competition and use the set as the logo." I was hooked.

In a flurry of phone calls the basic structure of the competition rapidly took shape. Richard wrote Benoit Mandelbrot and obtained permission to use his name for our contest. He must have been flattered, because he agreed; we still have his response stored with other Mandelbrot memorabilia. I suggested a team test in which students would write out proofs rather than fill in blanks. This idea was greeted enthusiastically by the newly elected Greater Testing Concepts board of managers, since mathematical writing was a component we felt was sorely lacking in contests while we were high school students. A few years later Sandor would make the brilliant observation that our contest could benefit from two divisions of differing difficulty, bringing the competition to its present form.

This book compiles every team test, solution, and essay used for the Mandelbrot Competition during its first five years of existence, with a few modifications. The layout has been reformatted slightly to accommodate a spiral bound book. The phrasing of some questions has been altered to render them more clear. The solutions and essays have been extensively edited to improve their readability and mathematical content, and an index has been included to reference test questions by topic. We chose the spiral binding so that pages could be easily removed and copied as needed. Coaches of currently registered Mandelbrot teams are welcome to duplicate materials for team preparation, up to twenty copies per page per year.

There is a tremendous amount of material contained in these pages, much more than can be assimilated in several weeks or even months. Browse through the tests or glance at the index to get a feel for the types of topics covered. Pick a test which looks interesting and work on it. The A division tests are in general more difficult than the B division tests, so peek at B tests for hints, or try A tests for a challenge. The tests from the first two years, before the different divisions were created, are about as difficult as the A tests from later years. If you get stuck, DON'T GIVE UP! Bounce ideas off of a classmate, or simply put the test down and come back to it the next day. Read half of the solution and then try to finish the rest of the question on your own. Learn a bit more about the topic by consulting the appropriate chapters of *The Art of Problem Solving*, a terrific two volume introduction to all of these topics and more, available through Greater Testing Concepts. But most of all, enjoy yourself. We hope that the results developed on these tests help to introduce you to some of the beautiful aspects of mathematics which are well within your reach as a high school student.

If you would like to share a creative alternate solution, or obtain permission to develop some team test idea further as part of mathematical paper or project, or point out a mathematical or typographical error in this volume, please do not hesitate to write Sam Vandervelde, P.O. Box 380789, Cambridge, MA 02238-0789. For more information on joining the Mandelbrot Competition or to order copies of *The Art of Problem Solving* please visit our website at `http://www.mandelbrot.org`, send us email at `info@mandelbrot.org`, or write to Greater Testing Concepts, P.O. Box 380789, Cambridge, MA 02238-0789.

Finally, I wish to thank Richard Rusczyk and Sandor Lehoczky for inviting me to join them on this fantastic venture. They have inspired me to be more creative than I could have ever imagined, and they have been supportive partners as well as good friends. Without Richard's resourcefulness the Mandelbrot Competition would not have reached half the number of students it has, and without Sandor's vision we would have never incorporated LaTeX or the world wide web into this competition, to name just a few of his contributions. I thank Eunice Cheung as well as Richard and Sandor for taking the time to listen to my ideas, suggest many more of their own, and proofread this text. My gratitude also to Bernice Cheung for creating the sketches that appear throughout these pages. Thank-you.

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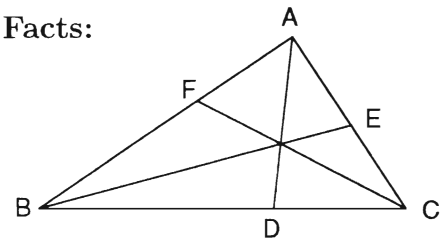
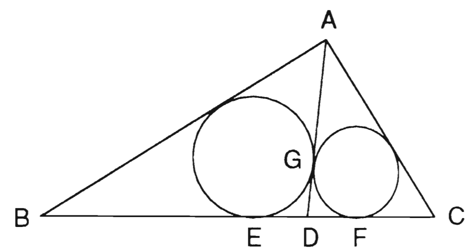


Points D, E, and F are on the sides of triangle $\triangle ABC$. *Ceva's Theorem*: Lines AD, BE, and CF are concurrent if and only if

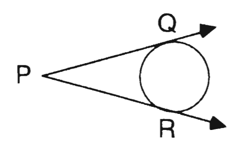
$$(AF)(BD)(CE) = (BF)(CD)(AE).$$

Diagram: The two incircles (circles tangent to all three sides of a triangle) are tangent to BC at E and F. Both circles are tangent to AD at G.



Problems:

Part i: Let P be a point outside a given circle. There are two lines through P which are tangent to the circle. If Q and R are the points of tangency, show that $PQ = PR$.



Part ii: Suppose that point D is chosen on side BC of triangle $\triangle ABC$ such that the incircles of $\triangle ABD$ and $\triangle ACD$ are each tangent to $\overline{AD}$ at the same point G, as in the diagram above. If we label $AB = c$, $AC = b$, and $BC = a$, then find the length of BD in terms of a, b, and c.

Part iii: Let the radii of the two circles in the diagram be r and s. Show that the length of DF is $\sqrt{rs}$.

Part iv: Let line l be the angle bisector of angle $\angle ABC$, line m be the angle bisector of angle $\angle ACB$, and line n be the perpendicular to BC at point D. Prove that lines l, m, and n are concurrent by showing that all three lines pass through the center of the incircle of $\triangle ABC$.

Part v: Suppose that in $\triangle ABC$, points H and J are defined on segments AC and AB in the same manner that D was defined on segment BC in part ii. Using Ceva's theorem, prove that lines AD, BH, and CJ are concurrent.

December 1990

The Mandelbrot Competition

Round Two Team Test

Facts: The symbol $\binom{n}{k}$ represents the number of ways to choose k blocks from a row of n blocks. For example, we find $\binom{3}{2} = 3$ since from a row of three blocks we can exclude either the first, second, or third block and pick the other two, for a total of three ways to choose two blocks from a row of three. The numerical value of $\binom{n}{k}$ is given by the formula

$$\binom{n}{k} = \frac{n!}{(n-k)!k!},$$

where $n! = (n)(n-1)\cdots(2)(1)$. By convention we set $0! = 1$.

Problems:

In the next three parts we are choosing n blocks from a row of $n+k$ blocks, $k \geq 1$.

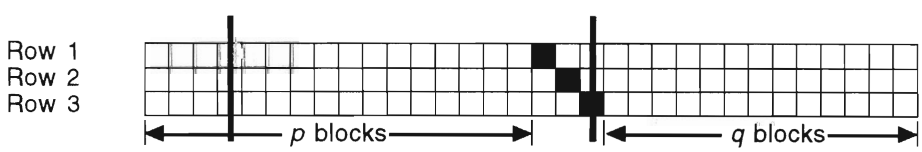
Part i: Within this row of $n+k$ blocks, show that there are $\binom{n+k-1}{n-1}$ groups of n blocks which include the first block.

Part ii: In general, show that there are $\binom{n+k-m}{n-1}$ groups of n blocks which include the m^{th} block, but no blocks to the left of the m^{th} block.

Part iii: We can classify each group of n blocks by the position of the leftmost block in the group. Use this idea to prove the binomial identity

$$\binom{n-1}{n-1} + \binom{n}{n-1} + \cdots + \binom{n+k-1}{n-1} = \binom{n+k}{n}.$$

Part iv: Now suppose that we are given three rows of blocks, each of length $p+q+3$, with three shaded blocks on a diagonal, as shown below.



L	R
0	2
1	1
2	0

Choose any two columns. The numbers to the right of the rows predict how many chosen columns will be to the left (L) and right (R), but not on top of, the shaded block in that row. Prove that *exactly one* of the predictions is correct. As an example, in the above diagram only the prediction for the middle row is correct.

Part v: Use part iv to prove the following binomial identity:

$$\binom{q+2}{2}\binom{p}{0} + \binom{q+1}{1}\binom{p+1}{1} + \binom{q}{0}\binom{p+2}{2} = \binom{p+q+3}{2}.$$

January 1991

The Mandelbrot Competition

Round Three Team Test

Facts: The *arithmetic mean* of a set of n positive numbers is the sum of the numbers divided by n. The *geometric mean* is the n^{th} root of the product of the numbers. For example, the arithmetic mean of the set $\{1,3,9\}$ is $\frac{1}{3}(1+3+9)=4\frac{1}{3}$, while the geometric mean is $\sqrt[3]{(1)(3)(9)}$, or 3. The arithmetic mean of any set of positive numbers is greater than or equal to the geometric mean of the set. The two means are equal if and only if all the numbers in the set are equal. This inequality is commonly referred to as AM-GM.

The value of $\min\{a,b,c\}$ is the minimal value of a, b, and c. Similarly, $\max\{a,b,c\}$ is the maximal value of a, b, and c. The easily misunderstood notation $\min\max\{a,b,c\}$ denotes the following: for each possible triplet $\{a,b,c\}$ choose the largest element. Then $\min\max\{a,b,c\}$ is the smallest of all these maximums. The value of $\max\min\{a,b,c\}$ is defined similarly. For example, if a, b, and c are different integers less than 10, then $\max\min\{a,b,c\}$ is 7, since the largest possible minimum occurs when $\{a,b,c\}$ is $\{7,8,9\}$.

Problems:

Part i: For positive x, y, and z such that $xyz=1$, use the arithmetic-geometric mean inequality to show that $\min\max\{x+y,y+z,z+x\}=2$.

Part ii: Again, let $xyz=1$ with x, y, and z positive. Show that $\min\{x+y,y+z,z+x\}$ can be as large as we wish for suitable x, y, and z.

Part iii: For positive x, y, and z such that $x+y+z=3$, show that $\max\min\{xy,xz,yz\}=1$.

Part iv: Show that for two positive real numbers a and b we have

$$\frac{a+b}{2}-\sqrt{ab}\geq\sqrt{\frac{a^2+b^2}{2}}-\frac{a+b}{2}$$

by showing that this inequality is equivalent to

$$\frac{(a+b)^2}{2}\geq\sqrt{(2ab)(a^2+b^2)}$$

and then using AM-GM.

Part v: Show that for positive a and b,

$$\sqrt{\frac{a^2+b^2}{2}}-\frac{a+b}{2}\geq\sqrt{ab}-\frac{2}{1/a+1/b}.$$

March 1991

The Mandelbrot Competition

Round Four Team Test

Facts: We commonly use the symbol $\mathbb{N}$ to represent the natural numbers, which are the numbers $\{1, 2, 3, \ldots\}$. If $f(m)$ is a function defined for all natural numbers, and $f(m)$ is itself a natural number in all cases, then we say that both the *domain* and *range* of f are the natural numbers, and we indicate this fact by writing $f : \mathbb{N} \to \mathbb{N}$. Unless stated otherwise, on this team test f and g will always refer to functions $f, g : \mathbb{N} \to \mathbb{N}$.

Given a function g, the function f is a solution to $f(g(m)) = g(f(m))$ if this equation holds true for all natural numbers in the domains of f and g. Finally, we say the function $f : \mathbb{N} \to \mathbb{N}$ is one-to-one and onto if for each natural number n there is *exactly one* natural number m such that $f(m) = n$.

Problems:

Part i: Show that for any g the function $f(m) = m$ is a solution to $f(g(m)) = g(f(m))$.

Part ii: If $g(m) = m^2$, describe all possible functions $f : \mathbb{N} \to \mathbb{N}$ that are solutions of

$$f(g(m)) = g(f(m)).$$

Part iii: Suppose that the domains and ranges of both f and g are restricted to the natural numbers $\{1, 2, \ldots, 7\}$. That is, $f(m)$ and $g(m)$ are only defined for $x = 1, 2, \ldots, 7$ and may only take on the values $\{1, 2, \ldots, 7\}$, though not necessarily all of them. How many distinct functions $f(m)$ exist such that $f(g(m)) = g(f(m))$ if $g(m)$ is defined as

a)

m	=	1	2	3	4	5	6	7
$g(m)$	=	1	2	3	4	5	6	7

b)

m	=	1	2	3	4	5	6	7
$g(m)$	=	2	4	3	1	7	5	6

c)

m	=	1	2	3	4	5	6	7
$g(m)$	=	1	1	2	3	4	5	6

.

Part iv: If $g : \mathbb{N} \to \mathbb{N}$ is one-to-one and onto, prove there is a one-to-one, onto function $f : \mathbb{N} \to \mathbb{N}$, besides the trivial solution $f(m) = m$, which satisfies $g(f(m)) = f(g(m))$.

April 1991

The Mandelbrot Competition

Round Five Team Test

Facts: A *proper divisor* of a positive integer n is a smaller positive integer which exactly divides n. For instance, 2 is a proper divisor of every positive even integer except for 2. A *perfect number* is a positive integer whose proper divisors sum up to the original number. The first two perfect numbers are 6 and 28. Because one should not routinely believe everything that one reads, let us verify that 28 is in fact a perfect number. The proper divisors of 28 are 1, 2, 4, 7, and 14, and we find $1 + 2 + 4 + 7 + 14 = 28$ as claimed.

Definitions: We now define a *magical set.* A magical set is a group of three or more positive integers, not necessarily distinct, such that each number in the set exactly divides the sum of the remaining numbers. If these numbers have no common divisor except 1 we call the set a *primitive* magical set. Thus the set $\{1, 2, 6, 9\}$ is a primitive magical set since $1+2+6 = 9$ is a multiple of 9; $1+2+9 = 12$ is a multiple of 6; $1+6+9 = 16$ is a multiple of 2; and $2+6+9 = 17$ is a multiple of 1. The set $\{2, 4, 12, 18\}$ is magical but not primitive because all the numbers are divisible by 2.

Problems:

Part i: Show that the set $\{1, 1, 2, 4, ..., 2^n\}$ is magical for all $n \geq 1$.

Part ii: Show that all the proper divisors of a perfect number form a magical set.

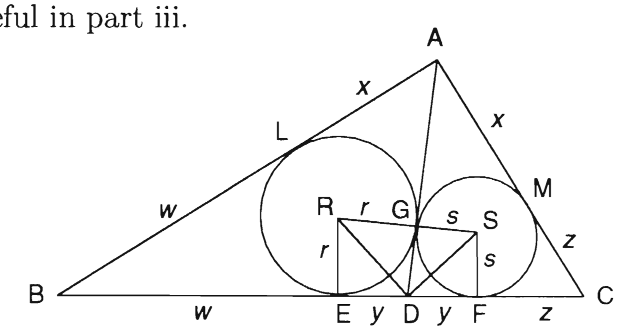
Part iii: Find all primitive magical sets with exactly three numbers.

Part iv: Find all magical sets with four numbers whose smallest elements are 1 and 3, i.e. of the form $\{1, 3, m, n\}$ with $m, n \geq 3$.

Part v: Prove that given any magical set, one can include an additional number in the set so that this new set is also magical.

Round One Team Test October 1990

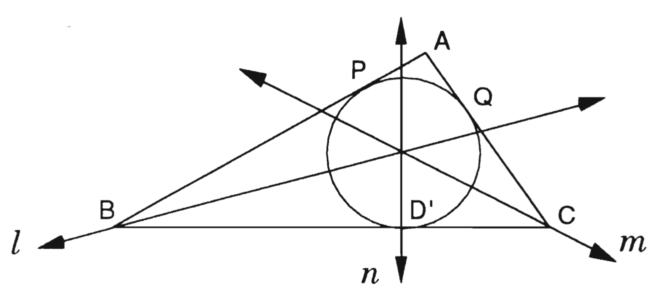
Part i: Let O be the center of the circle and draw radii $\overline{OQ}$ and $\overline{OR}$ which are equal in length. Since $\overline{PQ}$ and $\overline{PR}$ are tangents, angles $\angle PQO$ and $\angle PRO$ are right angles. By the hypotenuse-leg criteria, $\triangle PQO$ is congruent to $\triangle PRO$. Therefore corresponding sides $\overline{PQ}$ and $\overline{PR}$ are congruent, as desired. In addition, $\angle OPQ \cong \angle OPR$, so $\overrightarrow{PO}$ is the angle bisector of angle $\angle QPR$. This fact will be useful in part iii.



Part ii: Let L and M be the points where the two incircles are tangent to sides $\overline{AB}$ and $\overline{AC}$ respectively. From the previous proof we know that $BL = BE$; let us label this common distance w. Similarly we know that $AL = AG$ and that $AG = AM$. Hence $AL = AM$; we call this distance x. We also label $DE = DG = DF = y$ and $CF = CM = z$ as done in the diagram. We are looking for BD in terms of a, b, and c. Using the common distances just defined we have $a = w + 2y + z$, $b = x + z$, and $c = w + x$. Since we are seeking $BD = w+y$ we try $a+c-b = (w+2y+z)+(w+x)-(x+z) = 2(w+y)$. This computation proves that $BD = \frac{1}{2}(a - b + c)$.

Part iii: There are many approaches to this problem, most of them involving two observations. Let R and S be the centers of the left and right hand circles respectively. Then by part i the sum of angles $\angle RDG$ and $\angle SDG$ is half of the sum of angles $\angle EDG$ and $\angle FDG$ because $\overrightarrow{DR}$ and $\overrightarrow{DS}$ are angle bisectors. In other words, angle $\angle RDS$ is a right angle. Also, since $\overline{RG}$ and $\overline{SG}$ are both perpendicular to $\overline{AD}$, $\overleftrightarrow{RGS}$ is a straight line, so we have shown that $\triangle RDS$ is a right triangle. For convenience we label $RE = RG = r$ and $SF = SG = s$. We now apply the Pythagorean theorem repeatedly to find that $RD^2 = r^2+y^2$, $SD^2 = s^2+y^2$, and $RS^2 = RD^2+SD^2 = r^2+s^2+2y^2$. But $RS^2 = (r+s)^2 = r^2+2rs+s^2$. Equating the two expressions for RS^2 and canceling common terms we obtain $2y^2 = 2rs$, or $DF = y = \sqrt{rs}$.

Part iv: One of the advantages of proving a statement as opposed to calculating a number is that one knows in advance what the answer is, and can work backwards. We employ that strategy here. Namely, the two angle bisectors l and m meet at the incenter of $\triangle ABC$, so we know that the point of concurrency should be I, the incenter. If we can show that the perpendicular through D also passes through I we will be done. Notice that this will occur if D is the point of tangency of the incircle with side BC. Let D' be the actual point of tangency. We need to show that D and D' are the same point, which we will accomplish by showing that $BD = BD'$. Let P and Q



be the other points of tangency to the incircle as labeled in the diagram. Since tangents to a circle from the same point are congruent, we know $BD' = BP$, $AP = AQ$, and $CQ = CD'$. Using these equations we can write $a + c - b = BD' + D'C + BP + PA - AQ - CQ = 2BD'$. Hence $BD' = \frac{1}{2}(a + c - b)$, so $BD = BD'$. Thus D is in fact the point of tangency by virtue of being the proper distance from the vertex B, proving that all three lines are concurrent at the incenter.

Part v: Since $BD = \frac{1}{2}(a+c-b)$ and $BC = a$ we find that $CD = BC - BD = \frac{1}{2}(a+b-c)$. Defining the semiperimeter $s = \frac{1}{2}(a + b + c)$, we can write $BD = s - b$ and $CD = s - c$. If H and J are defined on $\overline{AC}$ and $\overline{AB}$ the same way that D was defined on $\overline{BC}$, then by symmetry we have $AH = s - a$, $HC = s - c$, $AJ = s - a$, and $BJ = s - b$. In this case it is clear that $(AH)(CD)(BJ) = (AJ)(BD)(CH)$, so $\overline{AD}$, $\overline{BH}$, and $\overline{CJ}$ are concurrent by Ceva's theorem.

Round Two Team Test — December 1990

Part i: Since the first block must be included in the group of n blocks, we have $n - 1$ blocks left to choose. These $n - 1$ blocks can be selected from any of the $n + k - 1$ remaining blocks in the row aside from the first one. By definition this can be done in $\binom{n+k-1}{n-1}$ ways.

Part ii: This result can be proved in a manner similar to part i. Once again we have $n - 1$ blocks to select, but only $n + k - m$ blocks left in the row from which we may select them, since none of the $n - 1$ blocks can be chosen from among the first m blocks. Again, by definition, this can be done in $\binom{n+k-m}{n-1}$ ways.

Part iii: First notice that part ii only makes sense if $m \leq k + 1$. For if $m > k + 1$ then the number of blocks left in the row from which we have to choose $n - 1$ blocks would be less than $(n + k) - (k + 1) = n - 1$, which won't work. We now divide up the $\binom{n+k}{n}$ groups of n blocks which can be chosen from the row of $n + k$ blocks. We label all groups which include the first block in the row of $n + k$ blocks with a **1**. Similarly, we label all groups which contain the second block but not the first with a **2**, and in general we label a group with an **m** if the first block of the group (from left to right) occurs in position m. Since $m \leq k + 1$, we have divided all $\binom{n+k}{n}$ groups into $k + 1$ categories. By part ii there will be $\binom{n+k-m}{n-1}$ groups labeled with an **m**, so there will be a total of

$$\binom{n+k-1}{n-1} + \binom{n+k-2}{n-1} + \cdots + \binom{n}{n-1} + \binom{n-1}{n-1}$$

groups in all the categories. Since we already know that there are a total of $\binom{n+k}{n}$ groups, we have shown that

$$\binom{n+k-1}{n-1} + \binom{n+k-2}{n-1} + \cdots + \binom{n-1}{n-1} = \binom{n+k}{n}.$$

Query: Can the reader prove this identity just using the fact that $\binom{n-1}{k-1}+\binom{n-1}{k}=\binom{n}{k}$?

Part iv: For simplicity, we will do a case by case analysis. A general proof can be constructed, but is significantly more difficult. The first possibility is that both columns lie to the right of the top shaded block. In this case no columns could be located to the left of the middle shaded block, and at most one column could be located to the left of the bottom shaded block, so only the top row's prediction would be correct. Suppose now that at least one column lies to the left of the middle shaded block. If the other column is located to the right of the middle shaded block, then only the middle row's prediction will be correct. Otherwise the second column is located to the left of the bottom shaded block, so only the bottom row's prediction will be correct. We have exhausted the possibilities for the placement of the two columns, and in every case we found that exactly one row's predictions was correct, so we are done.

Part v: As in part iii we will consider all possible pairs of columns that can be chosen: there are $\binom{p+q+3}{2}$ such pairs. Part iv indicates that each pair will fall into exactly one of three general categories: those pairs where both columns lie to the right of the top shaded block, those pairs which surround the middle shaded block, and those pairs where both columns are to the left of the bottom shaded block. In the first case we need to choose two columns among the $q+2$ columns to the right of the top shaded block and none from the p columns to the left of that block. This can be done in $\binom{p}{0}\binom{q+2}{2}$ ways. Similarly the second case can be done in $\binom{p+1}{1}\binom{q+1}{1}$ ways and the third in $\binom{p+2}{2}\binom{q}{0}$ ways (verify!). However, the number of pairs of columns in all the categories is $\binom{p+q+3}{2}$, so we have shown that

$$\binom{p}{0}\binom{q+2}{2}+\binom{p+1}{1}\binom{q+1}{1}+\binom{p+2}{2}\binom{q}{0}=\binom{p+q+3}{2}.$$

Query: Can the reader generalize parts iv and v to show that

$$\binom{p}{0}\binom{q+a}{a}+\binom{p+1}{1}\binom{q+a-1}{a-1}+\cdots+\binom{p+a}{a}\binom{q}{0}=\binom{p+q+a+1}{a}?$$

Round Three Team Test January 1991

Part i: It's a good bet that the arithmetic mean-geometric mean (AM-GM) inequality will be useful since we are dealing with sums, products, and inequalities. Keeping in mind that x, y, and z are positive with $xyz=1$, we write $\frac{x+y+z}{3}\geq\sqrt[3]{xyz}=1$, or $x+y+z\geq 3$. Returning to the original problem, we must find a way to show that the smallest a maximum can be is 2. This particular maximum can certainly be attained when $x=y=z=1$. Why can't it be lower? Let's suppose that the maximum was less than two for some choice of x, y, and z. Then $x+y<2$, $y+z<2$, and $z+x<2$. Adding these together yields

$2(x+y+z)<6$, or $x+y+z<3$, which is impossible by AM-GM. Therefore the smallest maximum is indeed 2.

Part ii: Upon experimenting one discovers that to obtain a huge minimum it suffices for two of the numbers to be very large, while the other one is so small that the entire product balances out to equal one. Let's make this precise. To show that the minimum can be arbitrarily large we choose any positive integer N. We then we let $x=N$, $y=N$, and $z=1/N^2$ which satisfies $xyz=1$ and yields $\min\{x+y, y+z, z+x\} = N+1/N^2 > N$. Thus the minimum of the pairwise sums can be made as large as we desire.

Part iii: This problem is similar to part i so one can prove it in an analogous manner, which the reader is encouraged to do. Instead, we present another argument which a student from Randolph High School submitted. By AM-GM we have the three inequalities

$$\frac{x+y}{2} \geq \sqrt{xy}, \quad \frac{x+z}{2} \geq \sqrt{xz}, \quad \frac{y+z}{2} \geq \sqrt{yz}.$$

Adding these together yields

$$3 = x+y+z \geq \sqrt{xy} + \sqrt{xz} + \sqrt{yz}.$$

This equation tells us is that it is impossible for all of $\sqrt{xy}$, $\sqrt{xz}$, and $\sqrt{yz}$ to be greater than 1, or else we would have $\sqrt{xy}+\sqrt{xz}+\sqrt{yz} > 3$. But $\sqrt{xy} > 1$ if and only if $xy > 1$, so the previous statement implies that it is impossible for all of xy, xz, and yz to be greater than 1. In addition, this minimum is realized when $x=y=z=1$. Hence $\max\min\{xy, yz, zx\} = 1$.

Part iv: The golden rule for proving inequalities is summed up by the word *reversibility*. Just because one is able to algebraically transform a given inequality into a true statement, such as one provable by AM-GM, doesn't mean that the original statement is true! Using this (incorrect) method one could argue, "I'll try to prove that $2 > 3$. Combining this equation with the fact that $3 > 1$ and using transitivity I get $2 > 1$, which is true. Therefore $2 > 3$." The key is to indicate how it is possible to work backwards from the valid inequality to the statement to be proved, i.e. reverse the steps. No difficulty arises when adding or multiplying to reach the next equation, since these operations have inverses. However, care must be taken when squaring both sides of an equation. The step $a \geq b \Rightarrow a^2 \geq b^2$ can be reversed if a and b are both positive, but not in general. Try $a=-3$ and $b=2$, for example.

With this strategy in mind, we simplify the given statement, watching out for sums and square-roots of products where an AM-GM may be applicable. The steps look like

$$\begin{array}{rrcll}
 & \frac{a+b}{2} - \sqrt{ab} & \geq & \sqrt{\frac{a^2+b^2}{2}} - \frac{a+b}{2} & \\
\Longleftrightarrow & a+b & \geq & \sqrt{\frac{a^2+b^2}{2}} + \sqrt{ab} & \text{(transposing terms)} \\
\Longleftrightarrow & a^2+2ab+b^2 & \geq & \frac{a^2+b^2}{2} + ab + 2\sqrt{\left(\frac{a^2+b^2}{2}\right)(ab)} & \text{(squaring both sides)} \\
\Longleftrightarrow & \frac{a^2+2ab+b^2}{2} & \geq & \sqrt{(2ab)(a^2+b^2)}. & \text{(collecting terms)}
\end{array}$$

At this point we realize that the final statement is true by AM-GM on the two numbers (a^2+b^2) and $(2ab)$. Since a and b are positive both expressions in the second step are also positive, so all steps are reversible and we have proved the claim. Notice that equality is achieved when $a^2 + b^2 = 2ab$, which is equivalent to $(a - b)^2 = 0$. Thus equality is attained only when $a = b$.

Part v: We proceed as before, writing

$$
\begin{array}{rrcll}
 & \sqrt{\frac{a^2+b^2}{2}} - \frac{a+b}{2} & \geq & \sqrt{ab} - \frac{2}{1/a+1/b} & \\
\iff & \sqrt{\frac{a^2+b^2}{2}} - \sqrt{ab} & \geq & \frac{(a-b)^2}{2(a+b)} & \text{(combining two fractions)} \\
\iff & \frac{(a+b)^2}{2} - \sqrt{(a^2+b^2)(2ab)} & \geq & \frac{(a-b)^4}{4(a+b)^2} & \text{(squaring both sides)} \\
\iff & \frac{2(a+b)^4-(a-b)^4}{4(a+b)^2} & \geq & \sqrt{(a^2+b^2)(2ab)} & \text{(combining fractions again)} \\
\iff & \frac{2(a+b)^4-(a-b)^4}{2} & \geq & \sqrt{(a^2+b^2)(2ab)(4(a+b)^4)}. & \text{(multiplying by } 2(a+b)^2\text{)}
\end{array}
$$

At this point we have a sum on one side and the square-root of a product on the other, but the numbers don't match up in order to use AM-GM. The difference of squares is suggestive, so we try

$$
\begin{array}{rrcl}
\iff & \frac{(a+b)^4+(a+b)^4-(a-b)^4}{2} & \geq & \sqrt{(a^2+b^2)(2ab)(4(a+b)^4)} \\
\iff & \frac{(a+b)^4+2(a^2+b^2)(4ab)}{2} & \geq & \sqrt{(2(a^2+b^2)(4ab))(a+b)^4}.
\end{array}
$$

We're in luck! This equation is true by AM-GM on the numbers $(a+b)^4$ and $2(a^2+b^2)(4ab)$. The only step which is not clearly reversible is the second in which we squared both sides of the inequality. It is left as a quick exercise for the reader to check that when a and b are positive then $\sqrt{(a^2+b^2)/2} - \sqrt{ab}$ is also. The right-hand side is clearly positive, so this step is also reversible, and we are done. Query: Can the reader show that equality is attained only when $a = b$?

Round Four Team Test — March 1991

First, a brief word about functions. A function is no more than a rule that associates an object from one set to an object in another set. I could create a function that pairs each year between 1980 and 1990 to the best selling math textbook of that year. Thus $f(1985) =$ *Functions For Everyone* by Nat Churrelog, or something like that. The domain of this function is any allowable input, in our case a year from 1980 to 1990. The range is allowable output, here the title of a math text book. The functions in all of the following problems associate two natural numbers, so the domain and range are $\{1, 2, 3, \ldots\}$, except in part iii.

Part i: Suppose that m is any natural number. Then $f(m) = m$ by our rule, so $f(m)$ and m are the same natural numbers, thus $g(f(m)) = g(m)$. Since $g(m)$ is also a natural number our rule $f(m) = m$ applies, and tells us that $f(g(m)) = g(m)$. But $g(f(m)) = g(m)$ also, so $f(g(m)) = g(f(m))$ for all natural numbers m.

Part ii: If $g(m) = m^2$, then $f(g(m)) = f(m^2)$ by substitution, and similarly $g(f(m)) = [f(m)]^2$. Hence we must describe all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(m^2) = [f(m)]^2$. Let's start experimenting. If $m = 1$ then $f(1) = [f(1)]^2$, so $f(1)$ is a natural number which equals its own square. The only natural number for which this is true is 1, so we must have $f(1) = 1$. What if $m = 2$? Then our equation tells us that $f(4) = [f(2)]^2$. So once $f(2)$ is decided upon, $f(4)$ is determined. However, there are no restrictions on $f(2)$, because 2 is not the square of any natural number! Similarly, $f(3)$ can also be defined arbitrarily. Next, $f(4) = [f(2)]^2$ as pointed out above, then $f(5)$ can be any natural number, and so on. In sum, the general solution is described by

- $f(1) = 1$,
- $f(m)$ can be defined arbitrarily whenever m is not a perfect square, and
- If m is perfect square, say $m = n^2$, then $f(m)$ is determined by the equation $f(m) = f(n^2) = [f(n)]^2$.

Note that the third requirement can be applied recursively for powers greater than two, for example $f(16) = [f(4)]^2 = [f(2)]^4$.

Part iii: a) Recall that $f(m) = m$ is a solution of $f(g(m)) = g(f(m))$ for any $g : \mathbb{N} \to \mathbb{N}$. Since $g(m) = m$ in this problem any function $f : \{1, 2, \ldots, 7\} \to \{1, 2, \ldots, 7\}$ is a solution using analogous reasoning. How many such functions exist? There are seven possible choices for each of $f(1)$, $f(2)$, $\ldots$, $f(7)$, for a total of $(7)(7)\ldots(7) = 7^7$ functions altogether.

b) A good way to attack these problems is to just plug in values of m. When $m = 1$ we find $g(f(1)) = f(g(1)) = f(2)$, since $g(1) = 2$. Trying the other six values for m leads to the seven equations shown below.

$$\begin{cases} f(2) = g(f(1)) \\ f(4) = g(f(2)) \\ f(1) = g(f(4)) \end{cases} \quad f(3) = g(f(3)) \quad \begin{cases} f(7) = g(f(5)) \\ f(5) = g(f(6)) \\ f(6) = g(f(7)) \end{cases}.$$

Once we choose a value for $f(1)$ then $f(2)$ is determined by the top left equation, which in turn dictates $f(4)$ using the next equation. The bottom left equation then yields $f(1)$, which should equal the value of $f(1)$ we already have if our choice of $f(1)$ is to lead to a solution. In fact, every one of the seven possible choices for $f(1)$ does work. The reader should try $f(1) = 3$ and $f(1) = 5$ to verify this fact. The same observations indicate that $f(5)$ can take on any value from 1 to 7, and that once $f(5)$ is chosen both $f(6)$ and $f(7)$ are determined. Finally, the middle equation shows that if $f(3)$ is the input for g then it remains unchanged. The only number fixed by g is 3, so we must have $f(3) = 3$. In sum, the values of $f(1)$ and $f(5)$ completely determine the solution, and each can take on seven distinct values, so there are 49 solutions.

c) We employ the same strategy as before, that is, we experiment. Suppose that we knew $f(7)$. The relation $f(g(m)) = g(f(m))$ when $m = 7$, along with the fact that $g(7) = 6$, implies $f(6) = g(f(7))$. In the same manner, using $m = 6$ in the general formula yields $f(5) = g(f(6))$, so we can determine $f(5)$ now that we know $f(6)$ from above. The same process continues on down to $f(1) = g(f(2))$ when $m = 2$, so once we choose $f(7)$ all values of f are determined. It remains to verify that these values constitute a solution.

By construction $f(g(m)) = g(f(m))$ for $m = 7, 6, \ldots, 2$. Hence we need only check that $g(f(1)) = f(1)$. It is an easy exercise to show that $f(m) \leq m$ for all m (can the reader find a quick demonstration?), so $f(1) \leq 1$ which means that $f(1) = 1$. Therefore $g(f(1)) = f(1)$ reduces to $g(1) = 1$ which is true, so the values of f yield a valid solution. For example, if we let $f(7) = 4$, then $f(6) = g(f(7)) = g(4) = 3$, $f(5) = 2$, and $f(4) = f(3) = f(2) = f(1) = 1$. Work this out on a piece of scratch paper and it will become clear! Since there are only seven choices for $f(7)$, this leads to just 7 solutions.

Part iv: One foolproof solution is the inverse of g. Recall that since g is one-to-one and onto there exists a unique m such that $g(m) = n$ for each natural number n, by definition. We exploit this fact to construct the inverse function f. To define $f(n)$, find the unique m such that $g(m) = n$, then set $f(n) = m$. Thus if g maps m to n, then f maps n back to m, or "inverses" the action of g. (The inverse of g is commonly denoted g^{-1}.) It follows that $f(g(m)) = m$ for all natural numbers m, and similarly $g(f(m)) = m$ also, so f is indeed a solution. There is one technical detail to attend to, namely, what if the above method produces the trivial solution $f(m) = m$? This scenario occurs only if $g(m) = m$ in the first place (verify!). It is a simple matter to produce a nontrivial solution in this case since any f is a solution by part i.

Several schools found a simpler and perhaps more obvious solution. They argued that $f(m) = g(m)$ is also a solution, since this would ensure that $f(m)$ was one-to-one and onto, and plugging this in yields $g(g(m)) = g(g(m))$, which is clearly true. The technical detail is handled as before.

Round Five Team Test — April 1991

Part i: The definition for magical sets is a little cumbersome as it is stated. It would be much easier to work with the sum of *all* the numbers in a magical set. Indeed, we note that in the set $\{1, 2, 6, 9\}$, the example given in the definition, the sum of all four elements is 18, and each element of the set divides 18. This observation motivates the following lemma.

LEMMA: *A set is magical if and only if each element in the set divides the sum of all the elements of the set.*

PROOF: Suppose that $M = \{a_1, a_2, \ldots, a_n\}$ is a magical set, and S is the sum of all its elements. Since M is magical, $a_2 + \cdots + a_n$ is a multiple of a_1. Therefore $(a_1) + (a_2 + \ldots + a_n) = S$ is also a multiple of a_1, so a_1 divides S. Since there was nothing special about using a_1 this reasoning will work equally well for any element of M, which shows that every element of M divides the sum S. Proving the converse requires no more than using the above arguments in the opposite order. Try outlining that proof for practice if this paragraph took more than a minute or so to digest.

The rest of part i follows readily. If $M = \{1, 1, 2, \ldots, 2^n\}$ then the sum of all the elements is $S = 1 + 1 + 2 + \ldots + 2^n = 2^{n+1}$. Clearly each element of M divides the sum S, so M is a magical set by the lemma. In fact M is a primitive magical set since one of the elements of M is 1.

Part ii: Let $M = \{d_1, d_2, \ldots, d_n\}$ be the set of proper divisors of some perfect number p. Then $p = d_1 + d_2 + \cdots + d_n$ by the definition of a perfect number, and each d_i is a divisor of p. Thus M is a set in which every element divides the sum of all the elements. Hence M is a magical set by the lemma. Again, 1 must be an element of M since 1 is a divisor of every perfect number, so M is in fact primitive.

Part iii: This proof is based on the solution submitted by Stuyvesant High School. Let the magical set be (a, b, c) listed in ascending order. Then $a \leq c$ and $b \leq c$ so $a + b \leq 2c$. But $a + b$ must be a multiple of c (it's a magical set), so either $a + b = c$ or $a + b = 2c$.

The latter case is easily disposed of, since the only way to achieve $a + b = 2c$ given $a \leq c$ and $b \leq c$ is if $a = c$ and $b = c$, yielding the set (c, c, c). We require the GCD to be 1 for a primitive magical set, so the only solution of this form is $(1, 1, 1)$.

We now assume that $a + b = c$. Since our set is magical $a + c = 2a + b$ is a multiple of b, which occurs if and only if $2a$ is a multiple of b. However, $a \leq b$ so $2a \leq 2b$, and the only multiples of b not greater than $2b$ are b and $2b$. Therefore either $2a = b$ or $2a = 2b$. In the first case we would have $c = a + b = 3a$, so our set becomes $(a, 2a, 3a)$. The only set of this form with GCD equal to 1 is $(1, 2, 3)$ since a is a common factor of all three elements. The second case proceeds similarly. One quickly sees that if $2a = 2b$ then our set is of the form $(a, a, 2a)$, yielding the only other possible three element primitive magical set, $(1, 1, 2)$.

Part iv: Our magical set is $\{1, 3, m, n\}$ where $m, n \geq 3$. We'll assume without loss of generality that $m \leq n$. Intuitively, the size of n is limited by the fact that it must be the largest element of the set while at the same time dividing the sum of the other three elements. We investigate this balance by writing $1 + 3 + m = kn$, using the fact that n divides the sum $(1 + 3 + m)$. Now

$$kn = 1 + 3 + m \leq 4 + n \quad \Rightarrow \quad k \leq 1 + \frac{4}{n} \leq 2\frac{1}{3},$$

since $n \geq 3$. Therefore we can only have $k = 1$ or $k = 2$.

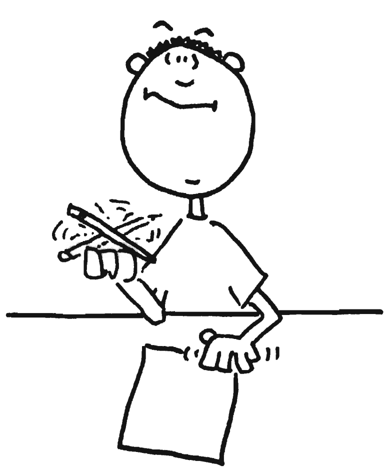
If $k = 1$ then our set becomes $\{1, 3, m, 4 + m\}$. Next, m also divides the sum of the other elements, so $m|(8 + m)$ which is equivalent to $m|8$. Since $m \geq 3$ either $m = 4$ or $m = 8$ which lead to the sets $\{1, 3, 4, 8\}$ and $\{1, 3, 8, 12\}$. Finally, checking that 3 divides the sum of the remaining elements reveals that only the second one is a magical set.

On the other hand, if $k = 2$ then $n = \frac{m+4}{2}$. Recalling that $3 \leq m \leq n$, it is easy to show that only $m = 4$ produces an integer n at least as large as m. Our set becomes $\{1, 3, 4, 4\}$ which is the only other solution.

Part v: One candidate for an additional element is the sum of the elements in the original set. If this sum is S, then the sum of the elements in the new set is clearly $2S$. By the lemma, we need only show that each element of the new set divides $2S$. Since the original set was magical every element divided S, and so will also divide $2S$. Furthermore, the additional element S divides $2S$. This accounts for all elements in the new set, which is therefore magical as well. Note that our new set is primitive if and only if the old one is.

1991-1992

The Second Year of the Mandelbrot Competition



Mandelbrot Morsels

Breakfast Mathematics

1991-92

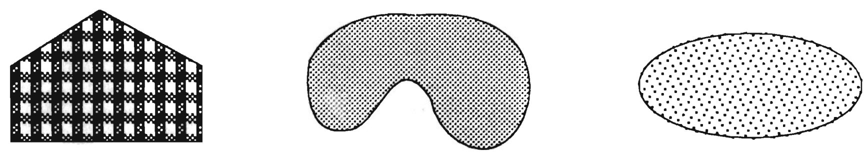
The purpose of this essay is to provide an introduction to a few concepts which, when combined with your own ingenuity, will solve all the problems on the round three team test. Along the way we will prove a fun theorem about figures in the plane informally known as the Pancake Theorem. Without further ado, here it is.

THEOREM: (The Pancake Theorem) Let A and B be two smooth convex figures in the plane. Then there exists a line which simultaneously bisects the areas of both figures.

In breakfast language that says, "It is possible to cut two (possibly overlapping) pancakes each in half with a single stroke of the knife."

Before diving into the proof, let's specify what is meant by a smooth convex figure. The modifier *smooth* pertains to the boundary of the figure and indicates that it has no sharp corners or angles but curves "smoothly," as illustrated by the second and third figures below, but not the first. The term *convex* technically means that the segment joining any two points of the region is contained entirely within the region. Intuitively this means that the region has no indentations. The first and third figures are convex, while the second is not. We restrict our attention to smooth convex figures so that no undesirable exceptions to our theorems will arise. This enables us to focus on the concepts without fussing with the technicalities.

Figures: 1 2 3

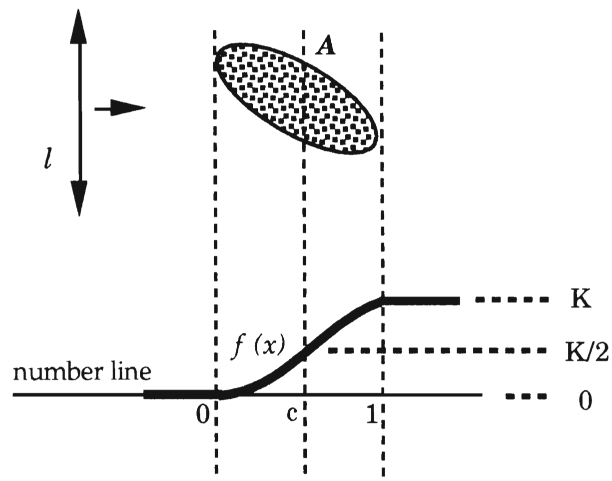


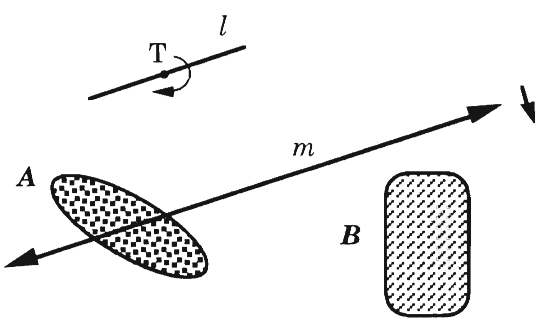
To build up to the theorem we will make use of a lesser result, a "lemma", which we will prove first. (Lemmas are sure signs of a good proof — I highly recommend them.)

LEMMA: If A is a smooth convex figure with area K and l is a given line, then there exists a unique line parallel to l which bisects the area of A.

PROOF: This is intuitively obvious. Translate the line l completely to the left of A (always keeping it parallel to its original position) so that all of the area is to the right of the line. Now slide the line across the figure until it is completely to the right of A, so that all of the area is to the left of the line. It stands to reason that at exactly one position in between half of the area was on each side of the line, so this is the desired unique line.

ALTERNATE PROOF: For the rigorously inclined, we will ground the above argument in some analysis. If you were happy with the first argument then please skip directly to the next paragraph. In order to make quantitative measurements we superimpose a number line perpendicular to l. For convenience's sake, assume that 0 lies just to the left of our figure A and scale the number line appropriately so that 1 is located just to the right of A. We next define a function $f(x)$ which measures the amount of area in A lying to the left of the number x. (More precisely, draw the line parallel to l through x; $f(x)$ measures how much area lies to the left of this line.) Then by our setup $f(0) = 0$ and $f(1) = K$, where K is the area inside A. Furthermore, $f(x)$ is a strictly increasing, continuous function for $0 \leq x \leq 1$. It is continuous because if we change x by a very small amount then area function $f(x)$ also only changes by a small amount. Since $f(0) = 0$ and $f(1) = K$ the Intermediate Value Theorem guarantees that $f(x)$ assumes all values between 0 and K as x varies from 0 to 1. Furthermore, the fact that $f(x)$ is strictly increasing guarantees that $f(x)$ achieves each value between 0 and K only once. In short, there is a unique number c between 0 and 1 such that $f(c) = \frac{1}{2}K$. This number corresponds to the unique line parallel to l which bisects the area of A.





The concept of translation was the key to the lemma. It turns out that rotation is the key to proving the pancake theorem.

Main Proof: Choose some line l to serve as an angular direction guide, and let m be the unique line parallel to l that bisects A. By some remarkable coincidence this line might also bisect B and we would be done, but this is not likely. We probably chose the wrong initial line, so rotate our direction indicator l a small amount around a swivel point T. Then line m will also rotate a little (it remains parallel to l). After line m rotates 180° it will return to its original position, since there is only one way to bisect figure A with line m when it is parallel to the initial direction. Notice that line m rotates in a continuous manner, always bisecting region A.

What has been happening with line m in relation to figure B all this time? Let the area of figure B be K', let x represent the amount of area in B lying above m, and let y be the amount of area below m. For example, in the diagram above $x = 0$ and $y = K'$. As m rotates through 180° we keep track of the quantity $x - y$, the difference between the area above and below m. Of course we would like this quantity to equal 0 at some point, because in that case line m would bisect the area of B. As pointed out before, after m has rotated 180° it returns to its original position. But now the difference between the top and bottom areas is $y - x$. (Just turn the diagram upside down; the top and bottom portions of figure B swap places.) Therefore as line m rotated the difference between top and bottom areas has changed continuously from $x - y$ to $y - x$. One of these is positive and the other is negative, so somewhere in between the difference must have been zero! Voila, the desired line bisecting both areas.

October 1991

The Mandelbrot Competition

Round One Team Test

Facts: The square of any integer leaves a remainder of either 0 or 1 when divided by 4. We prove this by considering two cases. If n is an even integer then it can be written as $2k$ for some integer k. Thus $n^2 = 4k^2$, which is divisible by four and hence leaves a remainder of 0. Otherwise n is odd and can be written as $2k+1$, so $n^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$, which clearly leaves a remainder of 1 when divided by 4. Similarly it may be shown that all squares leave a remainder of either 0 or 1 when divided by 3.

Definitions: If a set of one or more integers $\{a_1, a_2, \ldots, a_n\}$, not necessarily distinct, has the property that $\sum_{i\neq j} a_i a_j$ (the sum of the products of all pairs of integers in the set) is a perfect square, then we call such a set a *square set*. For example, the set $\{2, 3, 6\}$ is a square set since $2 \cdot 3 + 2 \cdot 6 + 3 \cdot 6 = 36$, a perfect square. We also associate a number b with a square set, where b is defined by

$$b = a_1 + a_2 + \cdots + a_n + 2\sqrt{\sum_{i\neq j} a_i a_j}.$$

In the example b would equal 23 since $2 + 3 + 6 + 2\sqrt{36} = 23$.

Problems:

Part i: Suppose that $\{a_1, a_2, \ldots, a_n\}$ is a square set, and b is defined as above. Show that the set $\{a_1, a_2, \ldots, a_n, b\}$ is also a square set.

Part ii: Let $\{a_1, a_2, \ldots, a_n\}$ and b be as in part i. Prove that

$$\{a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n\}$$

is also a square set. That is, b has the property that if any element of the original square set is replaced by b, then the new set is also a square set.

Part iii: If q is an integer (such as 4, 13, or 50) that can be expressed as the sum of two integer squares, show that both $2q$ and $5q$ can also be so expressed.

Part iv: If q is a positive integer that can be expressed as the sum of two integer squares, then prove that $3q$ cannot be so expressed.

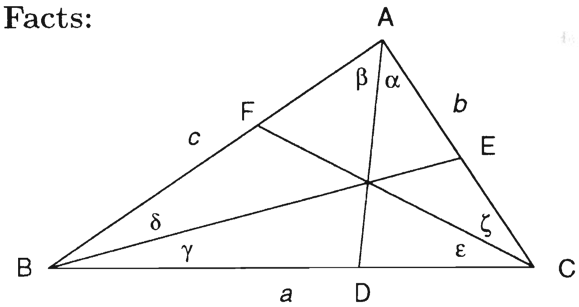
Part v: Prove that it is impossible for the squares of three consecutive integers to sum to another perfect square.

December 1991

The Mandelbrot Competition

Round Two Team Test

Facts:



Ceva's theorem states that cevians AD, BE, and CF are concurrent if and only if

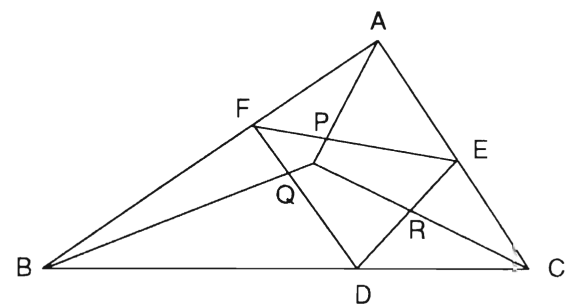
$$(AF)(BD)(CE) = (AE)(CD)(BF).$$

Another useful tool is the law of sines. If we label $AB = c$, $AC = b$, and $BC = a$ then

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c},$$

where A is the measure of angle $\angle BAC$ and similarly for B and C.

Diagram: Although not shown, lines AD, BE, and CF are concurrent, as are lines DP, EQ, and FR.



Problems:

Part i: Prove that cevians AD, BE, and CF are concurrent if and only if $(\sin\alpha)(\sin\delta)(\sin\epsilon) = (\sin\beta)(\sin\gamma)(\sin\zeta)$. The angles $\alpha, \beta, \ldots, \zeta$ are indicated on the diagram in the facts section.

Part ii: Let AD' be the line formed by reflecting line AD through the angle bisector of angle $\angle BAC$, and define BE' and CF' in an analogous manner. Prove that if AD, BE, and CF are concurrent then so are lines AD', BE', and CF'.

Part iii: Prove that the lines through A and the incenter of $\triangle ABC$, through B and the circumcenter of $\triangle ABC$, and through C and the orthocenter of $\triangle ABC$ are concurrent if and only if $\cos^2 A = \cos B \cos C$.

Part iv: Suppose that AD, BE, and CF are the altitudes of *acute* $\triangle ABC$, so that they lie properly within the triangle. Prove that in this case we have $\alpha = \gamma$, $\beta = \epsilon$, and $\delta = \zeta$, and conclude that the altitudes are concurrent.

Part v: Let D, E, and F be points on the sides of $\triangle ABC$ such that AD, BE, and CF are concurrent. Form triangle DEF, and select points P, Q, and R on its sides so that DP, EQ, and FR are concurrent, as in the diagram above. Prove that AP, BQ, and CR must also be concurrent.

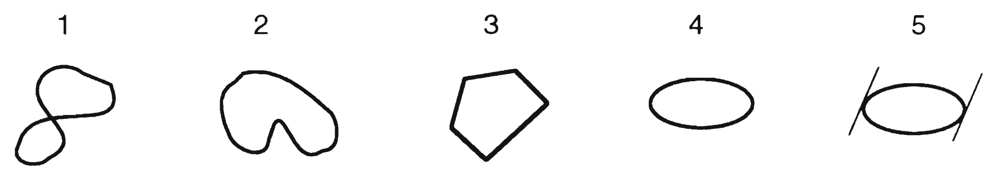
January 1992

The Mandelbrot Competition

Round Three Team Test

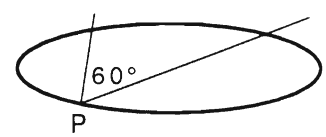
Definitions: A smooth convex figure is a convex figure whose boundary is smooth. A *smooth* boundary is one which has no sharp bends or angles, but curves "smoothly," such as in examples 1, 2, and 4 below, but not 3. Technically, a *convex* figure has the property that given any two points in the region, the straight line segment joining these two points also lies completely within the region. Intuitively this means that the figure has no indentations. Examples 3 and 4 below are convex, while examples 1 and 2 are not. There are exactly two tangents to the boundary of a smooth convex figure parallel to a given direction, one on each side, as pictured in example 5.

Examples:



Problems:

Part i: Let $\mathcal{A}$ be a smooth convex figure and let P be a given point on the boundary of $\mathcal{A}$. Prove that there exists a $60°$ angle with vertex at P enclosing exactly one-third of the area.

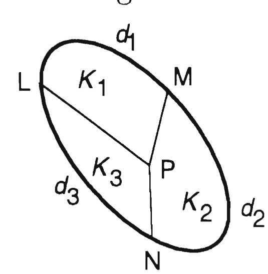


Part ii: Prove that it is possible to circumscribe a square about any smooth convex figure $\mathcal{A}$. This means that all four sides of the square are tangent to the boundary.

Part iii: Let $\mathcal{B}$ be a smooth convex figure whose boundary contains no line segments, i.e. always curves, as in examples 2 and 4 above, but unlike example 3. Let l be a given line. Prove that there is a unique line parallel to l which bisects the perimeter of $\mathcal{B}$, so that half of the boundary's length lies to one side of l, and the other half lies on the other side.

Part iv: Let $\mathcal{B}$ be a curve as in part iii. Prove that there exists some line m which simultaneously bisects the boundary of $\mathcal{B}$, as before, *and* bisects the area of figure $\mathcal{B}$.

Part v: Let $\mathcal{B}$ be a smooth convex figure containing no boundary segments as in part iii. Let the figure have perimeter d and area K. Suppose we choose three points L, M, and N on the boundary of $\mathcal{B}$ and a point P in the interior. Segments LP, MP, and NP divide the region into three areas K_1, K_2 and K_3. Points L, M, and N divide the perimeter into three lengths d_1, d_2, and d_3. Prove that it is possible to choose four such points L, M, N, and P so that



$$\frac{K_1}{d_1} = \frac{K_2}{d_2} = \frac{K_3}{d_3} = \frac{K}{d}.$$

March 1992

The Mandelbrot Competition

Round Four Team Test

Facts: Every positive integer n can be uniquely expressed as a sum of powers of three in the following manner: $n = a_k 3^k + \cdots + a_1 3 + a_0$, where each of the a_i is either 0, 1, or 2. This expression is commonly called the base three representation of the positive integer n. For example, the base three representation of 32 is $32 = 1(27) + 0(9) + 1(3) + 2(1)$, or simply $1012_{(3)}$ in shorthand notation.

On a somewhat different note, let α be a real number and let q be a positive integer. Then the sequence $\{\ldots, -\frac{1}{q}, 0, \frac{1}{q}, \frac{2}{q}, \ldots\}$ of rational numbers lies evenly spaced on the number line, and it is clear that α can be no farther than $\frac{1}{2q}$ from the nearest rational in this sequence. This fact can be stated compactly by saying that α can be approximated by a rational of the form $\frac{p}{q}$ with an accuracy of $\pm\frac{1}{2q}$. However, better approximations than this can be attained. A famous result from number theory states that given a real number α, there are infinitely many positive integers q such that α can be approximated by a rational of the form $\frac{p}{q}$ with an accuracy of $\pm\frac{1}{q^2}$. Finally, the fractional part of a positive number x is just the portion of that number following the decimal point and will be denoted $\{x\}$. For example, $\{2\} = 0$, $\{\pi\} = .14159265\ldots$, and $\{\frac{9}{8}\} = .125$. It may be useful to know that $\log_{10} 2 \approx .30103$.

Problems:

Part i: Show that every positive even integer n can be written

$$n = a_k 3^k + \cdots + a_1 3 + a_0,$$

where each a_i is either 0, 2, or 4.

Part ii: Show that *every* integer n has a "pseudo" base three representation in which each a_i is -1, 1, or 3. That is, n can be written $n = a_k 3^k + \cdots + a_1 3 + a_0$ with $a_i = -1$, 1, or 3.

In the following problems you will prove that infinitely many powers of three have 1 as their initial digit. All logarithms are base ten logarithms.

Part iii: Show that a positive integer m has initial digit 1 if and only if

$$0 \le \{\log m\} < \log 2.$$

Deduce that 3^k has initial digit 1 if and only if $0 \le \{k \log 3\} < \log 2$.

Part iv: Let α be a real number. Prove that there exist multiples of α that are arbitrarily close to an integer value.

Part v: Prove that infinitely many multiples of $\log 3$ have fractional part between 0 and $\log 2$. Conclude that infinitely many powers of three commence with the digit 1.

April 1992

The Mandelbrot Competition

Round Five Team Test

Facts: Suppose that points A, B, C, and D all lie on the same circle; then we call figure $ABCD$ a cyclic quadrilateral. Since an inscribed angle equals one half of its subtended arc, we know for instance that $\angle ABD \cong \angle ACD$, and that angles $\angle ABC$ and $\angle ADC$ are supplementary. Conversely, if B and C lie on the same side of line AD and $\angle ABD \cong \angle ACD$, then it follows that A, B, C, and D all lie on the same circle. Similarly, if B and D lie on opposite sides of $\overleftrightarrow{AC}$ and angles $\angle ABC$ and $\angle ADC$ are supplementary then points A, B, C, and D again lie on a single circle.

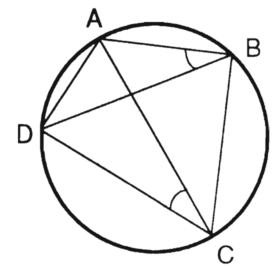
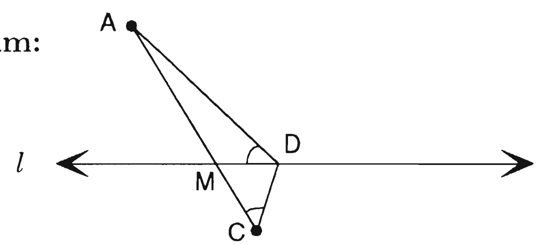


Diagram:



Points A and C are located on opposite sides of a given line l. Segment AC intersects line l at point M. Point D is located on l to the right of segment $\overline{AC}$ so that $\angle ACD \cong \angle ADM$.

Problems:

Part i: Given line l and points A, C, and M as shown in the diagram, prove that there is exactly one point D on l to the right of $\overleftrightarrow{AC}$ such that $\angle ACD \cong \angle ADM$.

Part ii: In the above diagram, let D be defined as before and let P be the unique point on l to the left of $\overleftrightarrow{AC}$ such that $\angle ACP \cong \angle APM$. Prove that $AP = AD$.

Part iii: Let Q be the unique point on l to the left of $\overleftrightarrow{AC}$ such that $\angle CAQ \cong \angle CQM$ (with D as before). Prove that $\overline{QC}$, $\overline{AC}$, and $\overline{AD}$ will form the sides of a right triangle.

Part iv: In the above diagram, what is the locus of point D as line l moves parallel to its initial position, always between points A and C?

Part v: Let $\overline{AC}$ be a given segment in the plane. What is the set of all points B in the plane, but not on line $\overleftrightarrow{AC}$, such that $m\angle ABC$ is greater than $m\angle BAC$?

Round One Team Test October 1991

Part i: Although the first two parts may seem like just a lot of algebra (think of it as good practice), when combined they prove the rather interesting assertion that given any square set there exists a positive integer which can either join the given set or replace any of its elements to create a new square set.

We know that the sum of the products of all pairs of numbers in the set $\{a_1, a_2, \ldots, a_n\}$ is a perfect square since this is a square set. Call this number x^2 and let the sum of all the elements be s. Then the definition of b can be rewritten as $b = s + 2x$. We must now check to see if the set $\{a_1, a_2, \ldots, a_n, b\}$ is a square set. The sum of the products of all pairs is

$$\begin{aligned}(a_1a_2 + a_1a_3 + \cdots + a_{n-1}a_n) + (a_1b + \cdots + a_nb) &= x^2 + sb \\ &= x^2 + s(s + 2x) \\ &= (x + s)^2.\end{aligned}$$

Sure enough, the result is a perfect square.

Part ii: This computation will be very similar to the last one. Let x^2 and s be as before, and suppose without loss of generality that a_1 is the element of the set being replaced by b. (The same argument will apply for any a_i, so we choose a_1 for convenience.) The sum of the products of all pairs in the new set is

$$\begin{aligned}& a_2a_3 + a_2a_4 + \cdots + a_{n-1}a_n + ba_2 + \cdots + ba_n \\ = \;& (a_1a_2 + a_1a_3 + \cdots + a_{n-1}a_n) - a_1(a_2 + \cdots a_n) + b(a_2 + \cdots + a_n) \\ = \;& x^2 - a_1(s - a_1) + b(s - a_1) \\ = \;& x^2 + (b - a_1)(s - a_1) \\ = \;& x^2 + (s + 2x - a_1)(s - a_1) \\ = \;& x^2 + 2x(s - a_1) + (s - a_1)^2 \\ = \;& (x + s - a_1)^2.\end{aligned}$$

Bingo, a perfect square.

Part iii: We are given an integer q which is the sum of two integer squares, so we can write $q = a^2 + b^2$. We now want to write $2q = 2a^2 + 2b^2$ as the sum of two squares. By introducing some extra terms we have

$$2q = a^2 + 2ab + b^2 + a^2 - 2ab + b^2.$$

It is now clear that $2q = (a + b)^2 + (a - b)^2$, proving the first half of the claim. How might one stumble onto this result? Experimenting with the algebra is one possible method; trying several examples and looking for a pattern is another. For example,

$$\begin{aligned}2(1^2 + 1^2) &= 0^2 + 2^2 \\ 2(1^2 + 2^2) &= 1^2 + 3^2\end{aligned}$$

$$2(1^2+3^2) = 2^2+4^2$$
$$2(1^2+4^2) = 3^2+5^2,$$

and so on. It is now a little easier to find or guess the correct representation for $5q$. Proceeding as above we discover that

$$5q = 5(a^2+b^2) = 4a^2+4ab+b^2+a^2-4ab+4b^2 = (2a+b)^2+(a-2b)^2,$$

so $5q$ can also be written as a sum of two squares.

Part iv: We employ the method of infinite descent. Suppose that it is possible for both q and $3q$ to be the sum of two integer squares. Then we could write $q = a^2+b^2$ and $3q = c^2+d^2$. Recall that c^2 and d^2 leave a remainder of either 0 or 1 when divided by 3. Since c^2+d^2 is a multiple of 3 it must be the case that both c^2 and d^2 are multiples of 3 (try all four possibilities), which means both c and d are multiples of 3. Therefore we let $c = 3c'$ and $d = 3d'$. Substituting into the equation $3q = c^2+d^2$ we find that $3q = 9c'^2+9d'^2$, so that $q = 3c'^2+3d'^2$, which means that q is a multiple of 3. Taking advantage of this fact, we write $q = 3q'$, which simplifies the last formula to $q' = c'^2+d'^2$. Incorporating our substitutions into the very first equation yields $3q' = a^2+b^2$.

In summary, we have deduced that if $q = a^2+b^2$ and $3q = c^2+d^2$ then $q' = c'^2+d'^2$ and $3q' = a^2+b^2$, where $q' = q/3$ is an integer. But this brings us back to exactly where we started! Now both q' and $3q'$ are the sum of two integer squares. Hence we can repeat this argument over and over, each time dividing q by another factor of 3. We deduce that if q has the property that both q and $3q$ can be expressed as the sum of two squares, then $q/3^k$ also has this property (and in particular is an integer) for every positive integer k. But this is absurd; no positive integer has infinitely many factors of three.

Part v: Label the three consecutive numbers $n-1$, n, and $n+1$. The sum of their squares is $(n-1)^2+n^2+(n+1)^2 = 3n^2+2$. This number clearly leaves a remainder of 2 when divided by three, so it cannot be a perfect square, according to the facts section. Now that wasn't too hard, was it?

Round Two Team Test December 1991

Part i: We approach this problem in a systematic manner. Since $\sin\alpha$ appears in the problem, it makes sense to apply the law of sines to a triangle containing angle α. The most likely such triangle in the diagram is $\triangle ADC$. Hence we can write $\sin\alpha/CD = \sin C/AD$. In the same manner we can write equations involving each of the other angles β, δ, γ, ϵ, and ζ. What else do we know? Ceva's theorem states that AD, BE, and CF are concurrent if and only if $(AF)(BD)(CE) = (AE)(CD)(BF)$. Using the above equation we can solve for CD to find $CD = (AD\sin\alpha)/\sin C$, and similarly we can solve for the remaining five lengths. Substituting in for all six lengths we conclude that AD, BE, and CF are concurrent if and only if

$$\left(\frac{AD\sin\alpha}{\sin C}\right)\left(\frac{BE\sin\delta}{\sin A}\right)\left(\frac{CF\sin\epsilon}{\sin B}\right)=\left(\frac{AD\sin\beta}{\sin B}\right)\left(\frac{BE\sin\gamma}{\sin C}\right)\left(\frac{CF\sin\zeta}{\sin A}\right),$$

which reduces to $(\sin\alpha)(\sin\delta)(\sin\epsilon) = (\sin\beta)(\sin\gamma)(\sin\zeta)$ when the common factors are canceled.

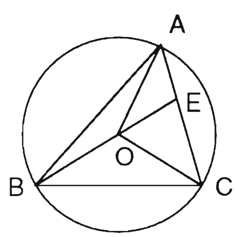
Part ii: We label angles $\angle CAD' = \alpha'$ and $\angle BAD' = \beta'$ for convenience. Since AM is the angle bisector of angle $\angle BAC$ it is clear that the image of line AB is line AC after a reflection over AM. In addition, the image of line AD is line AD' according to the hypotheses of the problem. Therefore the image of angle $\angle BAD$ is angle $\angle CAD'$. In particular, these two angles are congruent, so $\alpha' = \beta$. The same reasoning shows that $\beta' = \alpha$. Similar relationships hold between the angles formed at the other two vertices of $\triangle ABC$.

If AD, BE, and CF are concurrent then $(\sin\alpha)(\sin\delta)(\sin\epsilon) = (\sin\beta)(\sin\gamma)(\sin\zeta)$. Substituting the primed angles using the above equations yields

$$(\sin\beta')(\sin\gamma')(\sin\zeta') = (\sin\alpha')(\sin\delta')(\sin\epsilon').$$

By part i this is exactly what we need to conclude that AD', BE', and CF' are concurrent.

Part iii: In this proof we will adopt the common shorthand notation of referring to $m\angle A$ simply as A, and similarly for $m\angle B$ and $m\angle C$. In order to apply part i we need only compute each of the angles α, β, δ, γ, ϵ, and ζ in terms of A, B and C. The first two are simple; since line AD is the angle bisector of $\angle A$ we know that $\alpha = \beta = A/2$. Finding δ and γ is a little more complicated. Let O be the circumcenter of $\triangle ABC$ and draw segments AO, BO, and CO, each radii of the circumcircle. (Of course line BO is the same as line BE.) Then $m\angle BOA = 2C$ since $\angle BOA$ is a central angle subtending the same arc of the circumcircle as the inscribed angle $\angle BCA$. Also, $\triangle BOA$ is isosceles since both $\overline{AO}$ and $\overline{BO}$ are radii. Thus $\delta = m\angle OBA = \frac{1}{2}(180-2C) = 90-C$. By the same reasoning $\gamma = 90-A$. Finding ϵ and ζ is again fairly straightforward. Since $\overline{CF}$ is an altitude $\triangle CFB$ is a right triangle. Hence $\epsilon = m\angle BCF = 90-B$, and $\zeta = 90-A$ in an analogous manner.



Using the condition derived in part i for concurrency we find that AD, BE, and CF are concurrent if and only if

$$\sin(A/2)\sin(90-C)\sin(90-B) = \sin(A/2)\sin(90-A)\sin(90-A).$$

This equation is equivalent to $\cos^2 A = \cos B\cos C$ once we cancel the common factor and use the identity $\sin(90-x) = \cos x$.

Part iv: We have already encountered a similar situation in the previous problem, when we used altitude CF and right triangle $\triangle CFB$ to compute $\epsilon = 90-B$. Now AD is also an altitude, so right triangle $\triangle ADB$ implies that $\beta = 90-B$ as well, hence $\epsilon = \beta$. Chasing angles in the other right triangles demonstrates that $\alpha = \gamma$ and $\delta = \zeta$ as well.

The condition found in part i now follows trivially since it reduces to $(\sin\alpha)(\sin\delta)(\sin\epsilon) = (\sin\alpha)(\sin\delta)(\sin\epsilon)$ using the pairs of equal angles. Therefore the altitudes are concurrent.

Part v: How can we incorporate the hypotheses? Since AD, BE, and CF are concurrent we know by Ceva's theorem that

$$(AF)(BD)(CE) = (AE)(CD)(BF). \tag{1}$$

We are also given concurrent cevians DP, EQ, and FR so we know that

$$(DQ)(FP)(ER) = (DR)(EP)(FQ). \tag{2}$$

Triangles such as $\triangle APF$ and $\triangle APE$ seem like strategic sites for an application of the law of sines because their sides include lengths in both of the above equations. Hence we write $\sin\beta/FP = \sin(\angle APF)/AF$ and $\sin\alpha/PE = \sin(\angle APE)/AE$. Since $\angle APF$ and $\angle APE$ are supplementary their sines are equal. Dividing these two equations to eliminate this common quantity yields $\sin\beta/\sin\alpha = (FP/EP)(AE/AF)$. In the same way we can arrive at the corresponding equations $\sin\gamma/\sin\delta = (DQ/FQ)(BF/BD)$ and $\sin\zeta/\sin\epsilon = (ER/DR)(CD/CE)$. Multiplying all three equations together yields

$$\frac{\sin\beta\sin\gamma\sin\zeta}{\sin\alpha\sin\delta\sin\epsilon} = \frac{(FP)(DQ)(ER)}{(EP)(PR)(RD)} \cdot \frac{(AE)(BF)(CD)}{(AF)(BD)(CE)}.$$

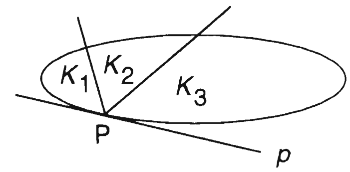
The right hand side of the above equation completely cancels using equations (1) and (2), so we are simply left with $(\sin\alpha)(\sin\delta)(\sin\epsilon) = (\sin\beta)(\sin\gamma)(\sin\zeta)$. But by part i this means that AP, BQ, and CR are concurrent as we wanted.

How does one know when to multiply and divide? Keep in mind that we want to use equations (1) and (2) to eventually cancel lengths and end up with something like part i. If this doesn't work, then just experiment! When the lengths don't cancel appropriately, perhaps multiplication was called for rather than division, or vice-versa.

Round Three Team Test — January 1992

As mentioned in the essay on the Pancake Theorem we are restricting ourselves to only smooth convex figures in these problems in order to avoid pathological counterexamples. For example, even the notion of length (intrinsic to bisecting a boundary) is tricky; there exist continuous bounded loops in the plane which have infinite length.

Part i: Draw the tangent line p to the curve $\mathcal{A}$ at point P. Next draw the two lines through P that make 60° angles with p (and thus with each other), as in the diagram. These three lines divide $\mathcal{A}$ into three regions. Let K be the total area enclosed by $\mathcal{A}$, and call the three smaller areas K_1, K_2, and K_3. If any of K_1, K_2, or K_3 is equal to $K/3$ then we are done. Otherwise one of K_1, K_2, or K_3 must be less than $K/3$, for if all were greater than $K/3$ we could conclude that $K_1 + K_2 + K_3 > K$, contradicting the fact that $K_1 + K_2 + K_3 = K$. Similarly, one of K_1, K_2, or K_3 is greater than $K/3$. Now smoothly rotate the 60° angle enclosing the area less than $K/3$ onto the 60° angle enclosing the area greater than $K/3$, keeping point P fixed throughout. The amount of area enclosed will vary continuously from less than $K/3$ to more than $K/3$, so at some point during the rotation a 60° angle will enclose exactly one-third of the area.



Part ii: Choose an arbitrary point P on $\mathcal{A}$, and draw the tangent line p to $\mathcal{A}$ at P. Also draw the second tangent parallel to p on the opposite side, and draw the two tangents on either side of $\mathcal{A}$ that are perpendicular to p. We now have a circumscribed rectangle; let a and b be the lengths of its sides with P on the side of length a. We construct a function f defined by $f(P) = b - a$ which measures the difference in side lengths of the rectangle based at a particular point of the curve. Observe that the rectangle is tangent to the curve at P and three other points. Let Q be one of the adjacent points of tangency and note that $f(Q) = a - b$ since Q lies on the side of length b. If $f(P) = 0$ then $a = b$ and the rectangle is a square, so we are done. Otherwise consider $f(R)$ as point R moves along the curve from P to Q. The function f ranges from $b - a$ at P to $-(b - a)$ at Q, so at some intermediate point $f(R) = 0$ and we obtain the desired square by basing it upon this point R.

Part iii: The proof of this assertion mirrors the area-bisecting lemma in the Pancake Theorem essay almost exactly, so I will only outline the details here. Translate the line l completely to the left of $\mathcal{B}$ so that the length of the perimeter to the right of the line is d. Now smoothly translate the line across the figure $\mathcal{B}$, always parallel to l, until none of the perimeter is to the right of the line. The length of the perimeter to the right of the translated line has decreased continuously from d to 0, so at exactly one position inbetween that length equaled $d/2$, yielding the unique line which bisects the perimeter.

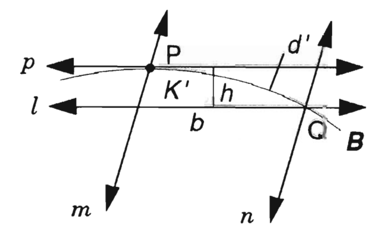
Notice that it is possible in some cases for the length of perimeter to jump discontinuously! For example, imagine a unit square in the Cartesian plane with vertices at $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$. As a line parallel to the y-axis moving to the right crosses the origin, the amount of perimeter to the right of the line jumps from 4 immediately to 3. This phenomena only occurs if part of the perimeter is a straight line segment, a possibility ruled out (precisely for this reason) by the hypotheses of the problem.

Part iv: Students at the Science Academy in Austin submitted a clever solution which I shall present here. Choose any point Q and designate a reference line q through Q. Then we know there are unique lines l_K and l_d parallel to q which bisect the area and perimeter respectively. If by some fantastic stroke of luck these lines coincide then we are done; otherwise assume for sake of argument that l_K is to the left of l_d. Now rotate line q about Q through 180° and observe the action of l_K and l_d. If these two lines never coincide then l_K remains on the same side of l_d throughout so that l_K winds up to the right of l_d after q has rotated 180°. But this is impossible because l_K and l_d must return to their original positions, since q returns to its starting position after a 180° rotation. Therefore at some point l_K and l_d coincide, and this is the desired line.

Part v: We first prove a slightly messy lemma.

LEMMA: Let P be a point on the smooth convex curve $\mathcal{B}$, let p be the tangent line at P, and let m be a second line through P distinct from p. Consider a line l parallel to p which intersects the region enclosed by $\mathcal{B}$. Lines l, m, and the curve $\mathcal{B}$ enclose a region with area K' whose perimeter includes a piece of length d' from $\mathcal{B}$, as illustrated below. Then as l

approaches p the ratio K'/d' tends to 0.

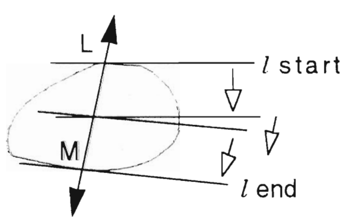


PROOF: The challenge in proving this lemma is figuring out a neat approach; the actual mathematics is simple. Let Q be the point of intersection of line l and the curve $\mathcal{B}$, and construct line n through Q parallel to m creating a parallelogram as shown at right. We also let b be the base and h the height of this parallelogram, so that its area is bh. Notice that this parallelogram surrounds the region of area K', so $K' < bh$. In addition we observe that $d' > h$ since the shortest distance between two lines is the perpendicular distance. Thus

$$\frac{K'}{d'} < \frac{bh}{d'} < \frac{bh}{h} = b,$$

and as l approaches p the distance b goes to 0, proving the lemma.

We will now construct points L, M, N, and P. The first step is to use part iv to choose a line that bisects both the perimeter and area. Label the points where this line intersects the curve L and M. If we choose P on $\overline{LM}$ then $K_1 = K/2$ and $d_1 = d/2$ so $K_1/d_1 = K/d$ as desired. All that remains is to choose P on $\overline{LM}$ and the point N somewhere on $\mathcal{B}$ between L and M so that $K_2/d_2 = K/d$, because then $K_3/d_3 = K/d$ automatically, as you can check.



To accomplish this we will move a line l continuously from being tangent to $\mathcal{B}$ at L to being tangent at M. One method of accomplishing this is to start with l tangent at L, translate l towards M until it reaches the midpoint of $\overline{LM}$, then rotate l about the midpoint until it is parallel to the tangent at M, and finish translating it down until it is actually tangent to $\mathcal{B}$ at M. Why is this useful? Keep track of the area and perimeter that are above l and to the right of $\overline{LM}$. By the lemma this ratio starts out very small, certainly less than K/d. But near M the ratio must be larger than K/d, because by the lemma the corresponding ratio of area to perimeter below l is very small. (Check the algebra as an exercise.) Thus the ratio is exactly K/d for some position of l along the way. This position determines the remaining two points P and N, completing the construction.

Round Four Team Test — March 1992

Part i: Since n is an even positive integer, $n/2$ is a positive integer. Thus $n/2$ has a base three representation

$$\frac{n}{2} = b_k 3^k + \cdots + b_1 3 + b_0, \qquad b_i = 0, 1, \text{or } 2.$$

Multiplying through by 2 yields

$$n = a_k 3^k + \cdots + a_1 3 + a_0, \qquad a_i = 0, 2, \text{or } 4,$$

where $a_i = 2b_i$. In other words, by doubling an integer we obtain an even integer, and doubling the base three representation for that integer yields a corresponding representation for the even integer.

Part ii: Once we allow digits other than 0, 1, or 2 in the base three representation of a number it becomes possible to write an integer in many different ways. For example, we can write $5 = 1(3) + 2(1) = 12_{(3)}$ as usual or write $5 = 2(3) - 1(1)$. After experimenting a bit the following observation becomes clear: if $n = a_k 3^k + \cdots + a_1 3 + a_0$ is a base three representation then we can create an alternate base three representation by replacing a_i by $a_i - 3$ and a_{i+1} by $a_{i+1} + 1$. The value of the sum doesn't change since we first decrease the sum by $3(3^i)$ and then increase it by 3^{i+1}, leaving the overall sum of n unchanged. Likewise we can also replace a_i by $a_i + 3$ and a_{i+1} by $a_{i+1} - 1$.

We can use this observation to alter a normal base three representation $n = a_k 3^k + \cdots + a_1 3 + a_0$ into one which only uses the digits -1, 1, and 3 by changing one digit at a time beginning with the one's place. If $a_0 = 0$ then increase a_0 to 3 and decrease a_1 by 1. If $a_0 = 1$ there is no need to alter this digit, and if $a_0 = 2$ then decrease a_0 by 3 down to -1 and increase a_1 by 1. The result of this algorithm is that a_0 is now equal to one of the desirable digits -1, 1, or 3, and a_1 equals one of the digits -1, 0, 1, 2, or 3. As before we need only change a_1 if it equals either 0 or 2, and we can proceed precisely as we did for a_0. Now a_1 is also either -1, 1, or 3, and a_2 lies between -1 and 3. Continuing this process for each digit will eventually produce the desired alternate base three representation.

A second proof is motivated by the fact that the coefficients -1, 1, and 3 are each one less than the coefficients 0, 2, and 4 needed to represent an even positive integer. This suggests writing n as the difference of an even integer (written as in part i) and a base three number $11\ldots1_{(3)}$ with the same number of digits. We leave the curious reader the task of carrying out this idea. Query: Can the reader discover more such triples of integers besides $\{-1, 1, 3\}$ that work in the same way? For instance, every integer can also be written as a sum of powers of 3 with coefficients $\{-1, 0, 7\}$. (Try to prove this!) With a little research this topic might turn into a fun paper.

Part iii: The primary challenge in this problem is setting it up clearly. For this reason we use "scientific notation" to write m in the form $m = x10^n$ where n is a positive integer and x is a terminating decimal with $1 \leq x < 10$. For instance, we would write $3142 = 3.142 \times 10^3$.

With these conventions it follows that $0 \leq \log x < 1$ and $\log m = n + \log x$, so that $\{\log m\} = \log x$. Thus $0 \leq \{\log m\} < \log 2$ is equivalent to $0 \leq \log x < \log 2$, which occurs if and only if $1 \leq x < 2$, i.e. when x has initial digit 1. But x and m clearly have the same initial digit since they differ by a multiple of 10^n. Thus $0 \leq \{\log m\} < \log 2$ if and only if m has initial digit 1, Which Was What We Wanted (W^5). The second part of the question follows at once since $\log 3k = k(\log 3)$.

Part iv: The facts section tells us that any real number α can be approximated by a rational p/q to within an accuracy of $\pm 1/q^2$ for infinitely many positive integers q. This can be written $|\alpha - p/q| < 1/q^2$. Multiplying by q yields $|q\alpha - p| < 1/q$, which means the multiple $q\alpha$ is within $1/q$ of the integer p. Since this is true for infinitely many q we can find multiples of α arbitrarily close to an integer, since the quantity $1/q$ approaches 0 as q goes to infinity. This approximation theorem was proved by Dirichlet using the Pigeonhole Principle and is discussed in the classic book by Hardy and Wright, *The Theory of Numbers*.

Part v: Choosing $\alpha = \log 3$ and only considering multiples $q \log 3$ with $q > 10$ (so that $1/q < .1$) we conclude by part iv that infinitely many multiples of $\log 3$ lie within .1 of an integer. This occurs if and only if $0 \leq \{q \log 3\} < .1$ or $.9 < \{q \log 3\} < 1$, by the definition of $\{x\}$. Considering our result from part iii we would hope that the former case occurs infinitely often, but this is not immediately clear. One of the two cases must occur for infinitely many q. If it is the former, then proceed as in the next paragraph. Otherwise there are infinitely many q such that $.9 < \{q \log 3\} < 1$. Choose any one of them and call it q_0. Then choose a larger one q such that $\{q_0 \log 3\} < \{q \log 3\}$, that is, $q > q_0$ and $q \log 3$ is closer to an integer than $q_0 \log 3$. There are infinitely many such integers q since the multiples of $\log 3$ become arbitrarily close to integer values. Let $(q_1, q_2, q_3, \ldots)$ be a list of these positive integers. By construction $q_1 - q_0$ is a positive integer with $\{(q_1 - q_0)\log 3\} < .1$ (convince yourself of this!). In the same way $\{(q_i - q_0)\log 3\} < .1$ for all $i \geq 1$, and the positive integers $(q_1 - q_0, q_2 - q_0, q_3 - q_0, \ldots)$ are all distinct.

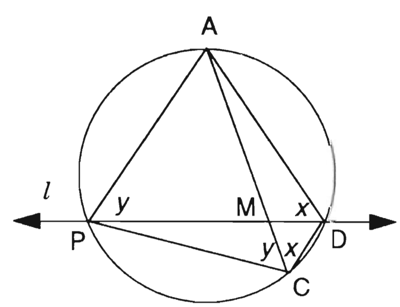
We have shown that in either case there are infinitely many positive integers q such that $0 \leq \{q \log 3\} < .1$. Since $\log 2 \approx .30103$ these multiples certainly lie within $\log 2$ of an integer. Therefore by part iii we conclude that infinitely many powers of three have initial digit 1.

Round Five Team Test — April 1992

Part i: The following solution was discovered by Chapel Hill and St. John's High School. Suppose that such a point exists. Then triangles $\triangle AMD$ and $\triangle ADC$ are similar because they share a common angle at vertex A and $\angle ADM \cong \angle ACD$ as well. Thus the sides have a common ratio, so $(AD/AM) = (AC/AD)$, or $AD^2 = (AM)(AC)$. In other words, AD has length equal to the geometric mean of AM and AC. But we know both these lengths, so we can also compute AD. Then we need only draw a circle through A with this constructed radius; this circle will intersect line l exactly once to the right of M. Therefore if point D exists it is unique.

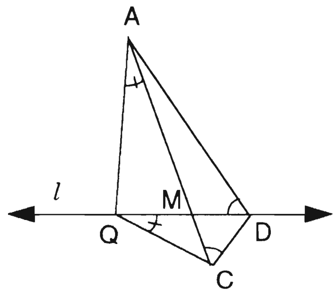
This construction also proves that such a point exists. If we construct D as described above then we can work backwards, using the common ratio to prove triangles $\triangle AMD$ and $\triangle ADC$ are similar, which shows that $\angle ADM \cong \angle ACD$. Hence exactly one point D fulfills the conditions of the problem.

Part ii: This result now follows quickly using the same type of argument given above. Can the reader fill in the details? We provide an alternate solution. Suppose that points P and D are given as described in the problem. Label $m\angle ADM = m\angle ACD = x$ and $m\angle APM = m\angle ACP = y$, as in the diagram. Then



$$m\angle PAD + m\angle PCD = (180° - x - y) + (x + y) = 180°,$$

from which we conclude by the facts section that $APCD$ is a cyclic quadrilateral. This is extremely useful, for now $\angle ADP$ and $\angle ACP$ both subtend the same arc of the circle, so they are equal. Hence $x = y$ which makes $\triangle APD$ isosceles, so that $AP = AD$.



Part iii: The hypotheses of the question yield the diagram at left, with equal angles marked. Just as in part i we can use similar triangles to deduce that $(AD)^2 = (AM)(AC)$. In an analogous manner we can prove that triangles $\triangle CMQ$ and $\triangle CQA$ are similar, using their common angle $\angle ACQ$ and the fact that $\angle CAQ \cong \angle CQM$. From the common ratios we can conclude as before that $(QC)^2 = (CM)(CA)$. Adding these equations yields

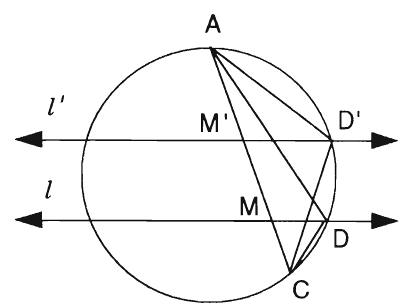
$$(AD)^2 + (QC)^2 = (AC)(AM) + (AC)(CM) = (AC)(AM + MC) = (AC)^2.$$

Therefore by the Pythagorean Theorem we can conclude that AD, QC, and AC would form the sides of a right triangle.

Part iv: It seems reasonable that the locus could be an arc of a circle. However, proving this guess is a two step process, much like an if and only if proof. To show that the locus is an arc of a circle one must first prove that if a point satisfies the conditions of the problem then it must be on the arc, and next also show that if a point is on the arc then it satisfies the conditions of the problem.

Recall that in the first problem we proved that $\triangle AMD$ and $\triangle ADC$ were similar, thus $\angle AMD \cong \angle ADC$. Now as line l moves parallel to its original position $\angle AMD$ does not change, so $\angle ADC$ remains constant as well. However, given fixed points A and C the collection of all points D to the right of l such that $\angle ADC$ is a given constant angle is an arc of a circle. (The proof of this theorem is based on nothing more than the results in the facts section. If it is unfamiliar to you I recommend a geometry text such as Coxeter and Greitzer's book *Geometry Revisited* from the Mathematical Association of America.) To construct this arc just circumscribe a circle about A, C, and the original point D. All points in the locus lie on minor arc $\overparen{ADC}$ between A and C.

To finish the other half of the problem suppose D' is an arbitrary point on minor arc $\overparen{ADC}$. Draw a line l' parallel to l through D' intersecting $\overline{AC}$ at M'. Then



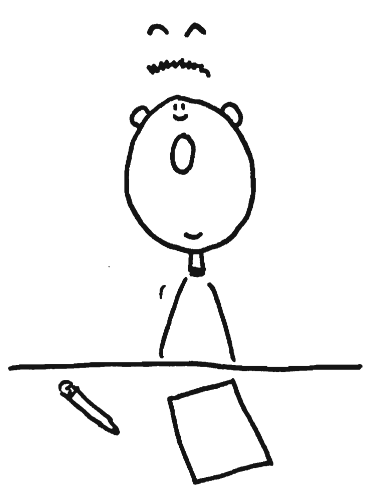
$$\angle AM'D' \cong \angle AMD \cong \angle ADC \cong \angle AD'C,$$

so that triangles $\triangle AM'D'$ and $\triangle AD'C$ are similar. Hence $\angle AD'M' \cong \angle ACD'$ which means that D' is part of the locus. Since D' was a generic point on arc $\overparen{ADC}$ all points of the arc are part of the locus.

Part v: An elementary theorem from plane geometry states that the longest side of a triangle is opposite the largest angle, while the shortest side is opposite the smallest angle. Therefore asking that $m\angle ABC > m\angle BAC$ is equivalent to requiring $AC > BC$. But given segment AC the set of all points in the plane such that $AC > BC$ is clearly the interior of the circle centered at C with radius AC. Hence the desired set consists of all points in the interior of this circle, excluding those on line AC.

1992-1993

The Third Year of the Mandelbrot Competition



Mandelbrot Morsels

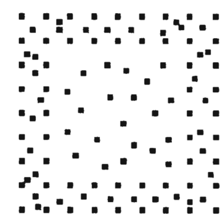
Sylvester's Theorem

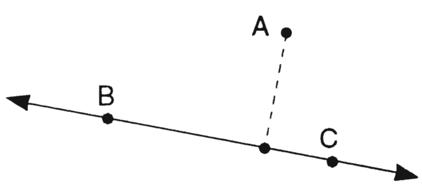
1992-93

The topic for team test two this year is combinatorial geometry, a fascinating field of study whose basics are easy to understand but likely are unfamiliar to the majority of high school students. This branch of mathematics combines elements of plane geometry (such as points, lines, distances, and areas) and combinatorics (counting techniques) while introducing some new strategies for solving problems. All will become clear during the following presentation of a classic theorem from combinatorial geometry. It is

THEOREM: (Sylvester's Theorem) Given n distinct points in the plane, not all collinear, then there exists a line that passes through exactly two of the points.

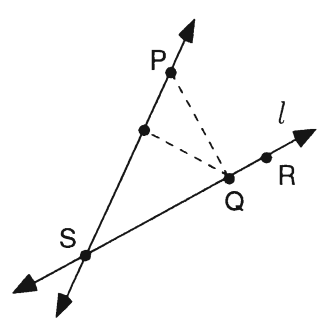
First, "not all collinear" means that the points don't all lie on the same line. (The claim wouldn't be true if all the points were collinear, except when there were only two points.) Sylvester's Theorem implies that it is impossible to situate a finite number of points in the plane in such a way that the line through any two of these points always passes through at least one other of our points. The problem is trivial if there are only two points, and still pretty clear if there are only three or four points. But how can we be sure that there exists a line through precisely two points in the diagram pictured at right, where $n = 112$? It turns out that the proof relies on a technique that I call the minimization principle, which basically means, "decide on some geometric quantity that can be measured (length, area, angle measure, etc.) and choose the configuration of points or lines which minimizes that quantity." With this principle in mind we begin the



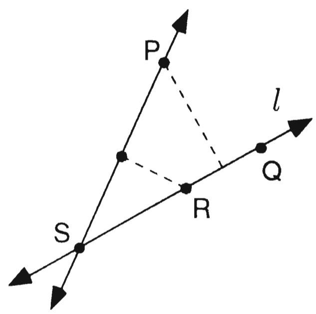


PROOF: Since the theorem is trivial for $n = 2$, we assume that $n \geq 3$. To any three points A, B, and C among the set of given points we associate the following geometric quantity: the distance from point A to the line through points B and C, indicated by a dashed line in the example shown to the left. Choose the set of three points (call them P, Q, and R) which have the smallest *positive* such distance. If two or more distances tie for the smallest positive distance, choose any one of them.

Since the points are not all collinear we can find at least one set of three points which form a triangle, so that at least one such distance is positive. As there are only a finite number of points there are only a finite number of ways to choose the three points P, Q, and R and consequently only a finite number of distances determined by the above process. Thus a smallest positive distance always exists.



How does this help? We claim that the line through Q and R (call it l) is the desired line! For suppose that there were another point S lying on l, and consider two cases. First, if one of the three points, say Q, is situated at the foot of the altitude from P to l then consider the distance from point Q to the line through P and S. Clearly this distance is less than the distance determined by points P, Q, and R, since the hypotenuse of a right triangle is longer than either of the legs. This contradicts the fact that we initially chose the minimum such distance.



Therefore none of the points can be situated directly at the foot of the altitude from P to l; so two of them, say R and S, lie on the same side of the foot of the perpendicular. Let R be the point closer to P, and consider the distance from R to the line through P and S. Exploiting the similar right triangles (the ones sharing common angle $\angle PSR$) we can show that this distance is smaller than the original one, again contradicting its minimality. We are forced to conclude that there could not have been a third point on l, proving Sylvester's theorem.

This is a very clever problem which is a bit too difficult to appear on a team test. The important thing to remember is the minimization principle – you will need this strategy to solve some (but not all) of the problems on team test two. Note that I took the liberty of skipping a few steps, such as the argument with the similar right triangles, and went overboard explaining how I set up the minimization principle. You can go through the set-up more quickly in your proofs, but don't omit any steps; prove all of your claims.

Best of luck on the team tests this year, and I hope you become interested enough in combinatorial geometry to look up some more theorems or maybe try to create some of your own. Just for fun, try to construct a counterexample to Sylvester's theorem using an infinite number of points.

November 1992

The Mandelbrot Competition

Division A Round One Team Test

Facts: The *Cauchy-Schwarz* inequality is named after the Frenchman and German who each formulated it independently at about the same time. The inequality states that given two sets of real numbers $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ then

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2.$$

Equality is achieved if and only if one set of numbers is a constant multiple of the other. In other words, there is a real number λ such that $b_1 = \lambda a_1$, $b_2 = \lambda a_2$, $\ldots$, and $b_n = \lambda a_n$; or vice versa, with $a_1 = \lambda b_1$ and so on.

Problems:

Part i: Prove the Cauchy-Schwarz inequality, including where equality is attained, in the case $n = 3$. (*Hint:* try $n = 2$ first.)

Part ii: Show that if α and β are angles in the first quadrant ($0° < \alpha, \beta < 90°$) then

$$\left(\frac{\cos^3\alpha}{\cos\beta} + \frac{\sin^3\alpha}{\sin\beta}\right)\cos(\alpha - \beta) \geq 1.$$

Part iii: Suppose that α and β are angles in the first quadrant. Prove that if

$$\left(\frac{\cos^3\alpha}{\cos\beta} + \frac{\sin^3\alpha}{\sin\beta}\right)\cos(\alpha - \beta) = 1$$

then $\alpha = \beta$.

Part iv: Let x_1, x_2, $\ldots$, and x_n be positive real numbers. Prove that

$$(x_1^{19} + x_2^{19} + \cdots + x_n^{19})(x_1^{93} + x_2^{93} + \cdots + x_n^{93}) \geq (x_1^{20} + x_2^{20} + \cdots + x_n^{20})(x_1^{92} + x_2^{92} + \cdots + x_n^{92}),$$

and find where equality holds.

Part v: At a wedding reception n guests have assembled into m groups of various sizes to converse. The host is preparing m square cakes, each with an ornate ribbon adorning its perimeter, to serve to the m groups. Due to dietary restrictions no guest is allowed to partake of more than 25 cm^2 of cake. Prove that no more than $20\sqrt{mn}$ cm of ribbon is needed to embellish the m cakes. (This is useful to know if you are buying the ribbon!)

November 1992

The Mandelbrot Competition

Division B Round One Team Test

Facts: The *Cauchy-Schwarz* inequality is named after the Frenchman and German who each formulated it independently at about the same time. The inequality states that given two sets of real numbers $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ then

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2.$$

Equality is achieved if and only if one set of numbers is a constant multiple of the other. In other words, there is a real number λ such that $b_1 = \lambda a_1$, $b_2 = \lambda a_2$, ..., and $b_n = \lambda a_n$; or vice versa, with $a_1 = \lambda b_1$ and so on.

Problems:

Part i: Prove the Cauchy-Schwarz inequality, including where equality is attained, in the case $n = 2$.

Part ii: Show that if α and β are angles in the first quadrant $(0° < \alpha, \beta < 90°)$ then

$$\left(\frac{\cos^3\alpha}{\cos\beta} + \frac{\sin^3\alpha}{\sin\beta}\right)\cos(\alpha - \beta) \geq 1.$$

Part iii: Suppose that α and β are angles in the first quadrant. Prove that if

$$\left(\frac{\cos^3\alpha}{\cos\beta} + \frac{\sin^3\alpha}{\sin\beta}\right)\cos(\alpha - \beta) = 1$$

then $\alpha = \beta$.

Part iv: Let x_1, x_2, ..., and x_n be positive real numbers. Prove that

$$(x_1^4 + x_2^4 + \cdots + x_n^4)(x_1 + x_2 + \cdots + x_n) \geq (x_1^2 + x_2^2 + \cdots + x_n^2)(x_1^3 + x_2^3 + \cdots + x_n^3),$$

and find where equality holds.

Part v: At a wedding reception n guests have assembled into m groups of various sizes to converse. The host is preparing m square cakes, each with an ornate ribbon adorning its perimeter, to serve to the m groups. Due to dietary restrictions no guest is allowed to partake of more than 25 cm^2 of cake. Prove that no more than $20\sqrt{mn}$ cm of ribbon is needed to embellish the m cakes. (This is useful to know if you are buying the ribbon!)

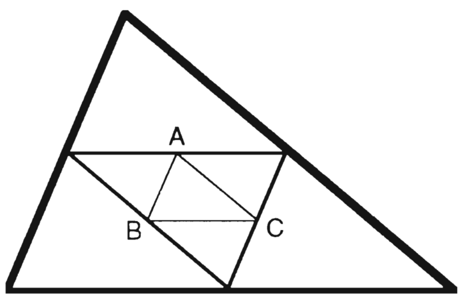
December 1992

The Mandelbrot Competition

Division A Round Two Team Test

Facts: Given n points in the plane, the number of ways to choose k of them is denoted $\binom{n}{k}$ and can be computed using the formula $\binom{n}{k} = \frac{n!}{(n-k)!k!}$, where $k! = k(k-1)\cdots(2)(1)$. For example, $\binom{5}{2} = \frac{5\cdot4\cdot3\cdot2\cdot1}{3\cdot2\cdot1\cdot2\cdot1} = 10$, so there are ten ways to choose a pair of points from among five given points in the plane.

Definitions: Given points A, B, and C, perform the following construction: through A, B, and C pass lines parallel to $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively. These three lines form a new triangle similar to $\triangle ABC$, whose sides are twice as long, and which has A, B, and C as the midpoints of its sides. Let's call this new triangle the *first outer medial triangle.* If we repeat this construction on the vertices of the new triangle we arrive at the *second outer medial triangle.*



Problems:

In all questions assume that no three of the given points are collinear.

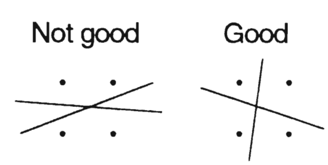
Part i: Show that among four points in the plane, no three of which form a right triangle, there exists at least one obtuse triangle.

Part ii: Now show that out of $n \geq 4$ points in the plane, no three forming a right triangle as before, there exist at least $\frac{1}{4}\binom{n}{3}$ obtuse triangles.

Part iii: Given n points in the plane, prove that one can find three of these points whose first outer medial triangle contains all remaining $n-3$ points in its interior or sides.

Part iv: Again, given n points in the plane, prove that one can find three points whose second outer medial triangle contains none of the other points in its interior.

Part v: Given $2n$ points in the plane, prove that one can find at least n lines, each dividing the points in half (n points on each side), such that different lines divide the points into different sets.

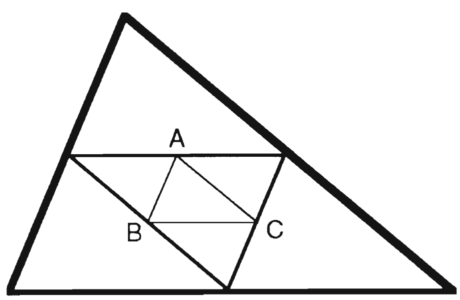


The Mandelbrot Competition

Division B Round Two Team Test

Facts: Given n points in the plane, the number of ways to choose k of them is denoted $\binom{n}{k}$ and can be computed using the formula $\binom{n}{k} = \frac{n!}{(n-k)!k!}$, where $k! = k(k-1)\cdots(2)(1)$. For example, $\binom{5}{2} = \frac{5\cdot4\cdot3\cdot2\cdot1}{3\cdot2\cdot1\cdot2\cdot1} = 10$, so there are ten ways to choose a pair of points from among five given points in the plane.

Definitions: Given points A, B, and C, perform the following construction: through A, B, and C pass lines parallel to $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively. These three lines form a new triangle similar to $\triangle ABC$, whose sides are twice as long, and which has A, B, and C as the midpoints of its sides. Let's call this new triangle the *first outer medial triangle.* If we repeat this construction on the vertices of the new triangle we arrive at the *second outer medial triangle.*



Problems:
In all questions assume that no three of the given points are collinear.

Part i: Show that among four points in the plane, no three of which form a right triangle, there exists at least one obtuse triangle.

Part ii: Suppose that $n \geq 4$ points in the plane are given, and k distinct triangles are designated, each with vertices among the n points. Show that no more than $k(n-3)$ of the $\binom{n}{4}$ groups of four points contain all three vertices of a designated triangle.

Part iii: There are $\binom{n}{3}$ triangles formed by $n \geq 4$ points in the plane, by definition. Suppose there are fewer than $\frac{1}{4}\binom{n}{3}$ obtuse triangles formed. Show that there is a group of four points which doesn't contain any of those obtuse triangles, and arrive at a contradiction by producing another distinct obtuse triangle. Conclude that at least one-fourth of all triangles formed by $n \geq 4$ points in the plane are obtuse.

Part iv: Given $2n$ points in the plane, show that there exist at least n different lines, each of which pass through two points in the set and divide the remaining $2n-2$ points in half ($n-1$ points on each side).

Part v: Given $n \geq 3$ points in the plane, prove that one can find three points whose second outer medial triangle contains none of the other points in its interior.

January 1993

The Mandelbrot Competition

Division A Round Three Team Test

Facts: Recall that the *degree* of a polynomial $p(x)$ is the exponent of the highest power of x, so the general form of a polynomial of degree n is $p(x) = a_n x^n + \cdots + a_1 x + a_0$. A polynomial of degree n is uniquely determined by its value at $n+1$ points. This means that the coefficients $a_0, a_1, \ldots, a_n$ are determined by the value of $p(x)$ at $n+1$ values of x. In particular, if two polynomials $p(x)$ and $q(x)$ of degree n agree for $n+1$ values of x then they must be *identically equal.* This relationship is written $p(x) \equiv q(x)$, which indicates that $p(x)$ and $q(x)$ have the same coefficients and agree for all values of x.

We also recall the ever useful *arithmetic mean-geometric mean* inequality (AM-GM). It states that the average of n positive real numbers is greater than or equal to the n^{th} root of their product.

Problems:

Part i: Consider the system of equations:
$$\begin{cases} a_1 + 8a_2 + 27a_3 + 64a_4 = 1 \\ 8a_1 + 27a_2 + 64a_3 + 125a_4 = 27 \\ 27a_1 + 64a_2 + 125a_3 + 216a_4 = 125 \\ 64a_1 + 125a_2 + 216a_3 + 343a_4 = 343 \end{cases}.$$

These four equations determine a_1, a_2, a_3, and a_4. Show that

$$a_1(x+1)^3 + a_2(x+2)^3 + a_3(x+3)^3 + a_4(x+4)^3 \equiv (2x+1)^3.$$

Part ii: The fact that the above two polynomials are identically equal yields several interesting relationships among the a_i. Deduce the following two equations:

$$a_1 + a_2 + a_3 + a_4 = 8 \quad \text{and} \quad 64a_1 + 27a_2 + 8a_3 + a_4 = 729.$$

Part iii: By considering a polynomial with roots $1, 1/2, \ldots, 1/n$ prove that

$$\frac{k}{1} + \frac{k}{2} + \cdots + \frac{k}{n} - \frac{k^2}{(1)(2)} - \cdots - \frac{k^2}{(n-1)(n)} + \cdots + (-1)^n \frac{k^n}{n!} = 1$$

for $k = 1, 2, \ldots, n$

Part iv: Let $r_1, r_2, \ldots, r_n$ be n positive real numbers. Prove that for any $x > 0$,

$$(x+r_1)(x+r_2)\cdots(x+r_n) \leq \left(x + \frac{r_1 + r_2 + \cdots + r_n}{n}\right)^n.$$

Part v: Again, let $r_1, r_2, \ldots, r_n$ be n positive real numbers. Prove that for any $x > 0$ we have $(x+r_1)(x+r_2)\cdots(x+r_n) \geq (x + \sqrt[n]{r_1 r_2 \cdots r_n})^n$.

January 1993

The Mandelbrot Competition

Division B Round Three Team Test

Facts: Recall that the *degree* of a polynomial $p(x)$ is the exponent of the highest power of x, so the general form of a polynomial of degree n is $p(x) = a_n x^n + \cdots + a_1 x + a_0$. A polynomial of degree n is uniquely determined by its value at $n+1$ points. This means that the coefficients a_0, a_1, ..., a_n are determined by the value of $p(x)$ at $n+1$ values of x. In particular, if two polynomials $p(x)$ and $q(x)$ of degree n agree for $n+1$ values of x then they must be *identically equal.* This relationship is written $p(x) \equiv q(x)$, which indicates that $p(x)$ and $q(x)$ have the same coefficients and agree for all values of x.

We also recall the ever useful *arithmetic mean-geometric mean* inequality (AM-GM). It states that the average of n positive real numbers is greater than or equal to the n^{th} root of their product.

Problems:

Part i: Consider the system of equations:
$$\begin{cases} a_1 + 8a_2 + 27a_3 + 64a_4 = 1 \\ 8a_1 + 27a_2 + 64a_3 + 125a_4 = 27 \\ 27a_1 + 64a_2 + 125a_3 + 216a_4 = 125 \\ 64a_1 + 125a_2 + 216a_3 + 343a_4 = 343 \end{cases}.$$

These four equations determine a_1, a_2, a_3, and a_4. Show that

$$a_1(x+1)^3 + a_2(x+2)^3 + a_3(x+3)^3 + a_4(x+4)^3 \equiv (2x+1)^3.$$

Part ii: The fact that the above two polynomials are identically equal yields several interesting relationships among the a_i. Deduce the following two equations:

$$a_1 + a_2 + a_3 + a_4 = 8 \quad \text{and} \quad 64a_1 + 27a_2 + 8a_3 + a_4 = 729.$$

Part iii: Prove that

$$\frac{n}{1} + \frac{n}{2} + \cdots + \frac{n}{n} - \frac{n^2}{(1)(2)} - \cdots - \frac{n^2}{(n-1)(n)} + \cdots + (-1)^n \frac{n^n}{n!} = 1$$

by considering the polynomial with roots 1, 1/2, ..., $1/n$.

Part iv: Let r_1, r_2, r_3, and r_4 be positive real numbers. Prove that for any $x > 0$,

$$(x+r_1)(x+r_2)(x+r_3)(x+r_4) \le \left(x + \frac{r_1+r_2+r_3+r_4}{4}\right)^4.$$

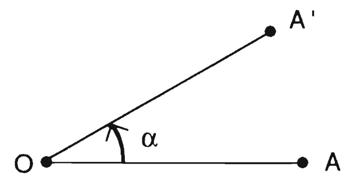
Part v: Again, let r_1, r_2, r_3, and r_4 be positive real numbers. Prove that for any $x > 0$ we have $(x+r_1)(x+r_2)(x+r_3)(x+r_4) \ge (x + \sqrt[4]{r_1 r_2 r_3 r_4})^4$.

March 1993

The Mandelbrot Competition

Division A Round Four Team Test

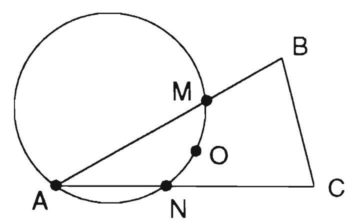
Facts: A *rotation* of the plane about a point O through an angle α maps the point O to itself and carries a point A to a new point A' such that $OA = OA'$ and $\angle AOA' = \alpha$. A rotation preserves distances: if a rotation maps points A and B to points A' and B' then $AB = A'B'$. A rotation also maps lines to lines.



If points A, B, C, and D (in that order) all lie on the same circle then we say these points are *cyclic* and we call figure $ABCD$ a *cyclic quadrilateral.* Properties of inscribed angles show that $\angle ABD = \angle ACD$ and $m\angle ABC + m\angle ADC = 180°$. Conversely, if B and C lie on the same side of line AD and $\angle ABD = \angle ACD$, then it follows that A, B, C, and D are cyclic. Similarly if B and D lie on opposite sides of AC and $m\angle ABC + m\angle ADC = 180°$, then points A, B, C, and D are again cyclic.

Problems:

Part i: Let $\triangle ABC$ be an isosceles triangle with $AB = AC$ and having circumcenter O. Prove that there exist rotations about both A and O which carry segment AB to segment AC.

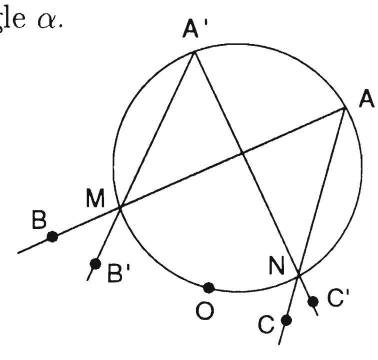


Part ii: Let $\triangle ABC$ be as above, and suppose points M and N on $\overline{AB}$ and $\overline{AC}$ are such that $BM = AN$. Prove that the circle through A, M, and N passes through O.

Part iii: Using this result, show that if $A_1A_2 \ldots A_n$ is a regular n-gon with an inscribed regular n-gon $B_1B_2 \ldots B_n$ (with B_1 on $\overline{A_1A_2}$, B_2 on $\overline{A_2A_3}$, ..., and B_n on $\overline{A_nA_1}$), then the circles through $\triangle B_1A_2B_2$, $\triangle B_2A_3B_3$, ..., and $\triangle B_nA_1B_1$ share a common point.

Part iv: Let line l be rotated to a new line l' about any point O through an angle α, $0° < \alpha < 180°$. Show that lines l and l' intersect in the angle α.

Part v: Suppose an angle $\angle BAC$ is rotated about a point O to a new angle $\angle B'A'C'$. Assume that the angle of rotation is between 0° and 180°. Let rays AB and $A'B'$ intersect at M, and rays AC and $A'C'$ intersect at N. Prove that A, A', M, N, and O all lie on a single circle. Notice that there are several possible diagrams.

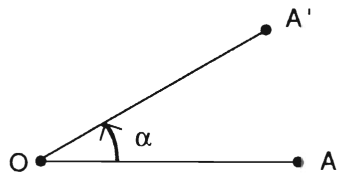


March 1993

The Mandelbrot Competition

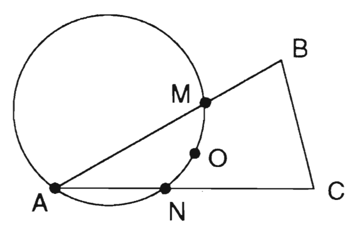
Division B Round Four Team Test

Facts: A *rotation* of the plane about a point O through an angle α maps the point O to itself and carries a point A to a new point A' such that $OA = OA'$ and $\angle AOA' = \alpha$. A rotation preserves distances: if a rotation maps points A and B to points A' and B' then $AB = A'B'$. A rotation also maps lines to lines.



If points A, B, C, and D (in that order) all lie on the same circle then we say these points are *cyclic* and we call figure $ABCD$ a *cyclic quadrilateral.* Properties of inscribed angles show that $\angle ABD = \angle ACD$ and $m\angle ABC + m\angle ADC = 180°$. Conversely, if B and C lie on the same side of line AD and $\angle ABD = \angle ACD$, then it follows that A, B, C, and D are cyclic. Similarly if B and D lie on opposite sides of AC and $m\angle ABC + m\angle ADC = 180°$, then points A, B, C, and D are again cyclic.

Problems:



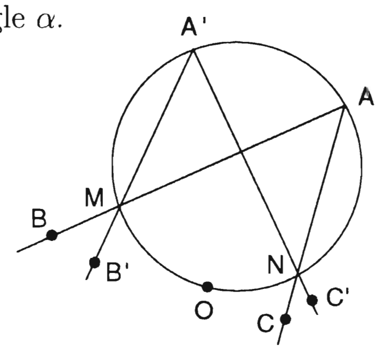
Part i: Let $\triangle ABC$ be an isosceles triangle with $AB = AC$, $m\angle BAC = \alpha$, and having circumcenter O. Prove that there exist rotations about both A and O which carry segment AB to segment AC.

Part ii: Let $\triangle ABC$ be as above, and suppose points M and N on $\overline{AB}$ and $\overline{AC}$ are such that $BM = AN$. Compute the angle of rotation (in terms of α) needed to map $\overline{AB}$ to $\overline{AC}$ with center O. Show that $m\angle MON$ equals this angle.

Part iii: Using this result show that the circle through A, M, and N passes through O.

Part iv: Let line l be rotated to a new line l' about any point O through an angle α, $0° < \alpha < 180°$. Show that lines l and l' intersect in the angle α.

Part v: Suppose an angle $\angle BAC$ is rotated about a point O to a new angle $\angle B'A'C'$, where the angle of rotation is between $0°$ and $180°$. Let rays AB and $A'B'$ intersect at M, and rays AC and $A'C'$ intersect at N. Prove that A, A', M, N, and O all lie on a single circle. Assume that the five points are situated as pictured in the diagram.



April 1993

The Mandelbrot Competition

Division A Round Five Team Test

Facts: The two systems of equations to the right are *equivalent;* in other words, each can be derived from the other.

$$\begin{cases} a+b+c = d+e+f & (1) \\ a^3+b^3+c^3 = d^3+e^3+f^3 & (2) \end{cases}$$

$$\begin{cases} (a+b)+(b+c)+(c+a) = (d+e)+(e+f)+(f+d) & (3) \\ (a+b)(b+c)(c+a) = (d+e)(e+f)(f+d) & (4) \end{cases}$$

For example, cube both sides of equation (1), subtract equation (2), and divide by three to obtain equation (4). Then multiply equation (1) by two to obtain equation (3). Working backwards, one can also derive equations (1) and (2) from equations (3) and (4). Because each system implies the other, equivalent systems of equations have exactly the same solutions. This is helpful if one set of equations is easier to work with than the other.

Problems:

Part i: Complete the argument started above by showing that equations (3) and (4) together imply equations (1) and (2).

Part ii: Find six *distinct* integers a, b, c, d, e, and f which satisfy equations (1) and (2) above. (In other words, trivial solutions such as $a = 0$, $b = 1$, $c = -1$, $d = 0$, $e = 2$, and $f = -2$ don't count.)

Part iii: Prove that it is possible to divide any eight consecutive integers into two sets such that the sum of the integers in each set is the same, and the sum of the squares of the integers in each set is also the same.

Part iv: Show that given any eight consecutive integers one can split them into two sets of four such that the sum of the cubes in one set subtracted from the sum of the cubes in the second set is a constant.

Part v: Building on the pattern begun above, prove the following theorem by induction.

THEOREM: *Given an integer $k \geq 1$ it is possible to divide any 2^{k+1} consecutive integers into two sets $\{a_1, a_2, \ldots, a_{2^k}\}$ and $\{b_1, b_2, \ldots, b_{2^k}\}$ such that*

$$\begin{aligned} a_1 + a_2 + \cdots + a_{2^k} &= b_1 + b_2 + \cdots + b_{2^k} \\ a_1^2 + a_2^2 + \cdots + a_{2^k}^2 &= b_1^2 + b_2^2 + \cdots + b_{2^k}^2 \\ &\vdots \\ a_1^k + a_2^k + \cdots + a_{2^k}^k &= b_1^k + b_2^k + \cdots + b_{2^k}^k \end{aligned}.$$

April 1993

The Mandelbrot Competition

Division B Round Five Team Test

Facts: The two systems of equations to the right are *equivalent;* in other words, each can be derived from the other.

$$\begin{cases} a+b+c = d+e+f & (1) \\ a^3+b^3+c^3 = d^3+e^3+f^3 & (2) \end{cases}$$

$$\begin{cases} (a+b)+(b+c)+(c+a) = (d+e)+(e+f)+(f+d) & (3) \\ (a+b)(b+c)(c+a) = (d+e)(e+f)(f+d) & (4) \end{cases}$$

For example, cube both sides of equation (1), subtract equation (2), and divide by three to obtain equation (4). Then multiply equation (1) by two to obtain equation (3). Working backwards, one can also derive equations (1) and (2) from equations (3) and (4). Because each system implies the other, equivalent systems of equations have exactly the same solutions. This is helpful if one set of equations is easier to work with than the other.

Problems:

Part i: Find six *distinct* integers a, b, c, d, e, and f which satisfy equations (1) and (2) above. (In other words, trivial solutions such as $a = 0$, $b = 1$, $c = -1$, $d = 0$, $e = 2$, and $f = -2$ don't count.)

Part ii: Prove that given four consecutive perfect squares, the sum of the second and third subtracted from the sum of the first and fourth is a constant.

Part iii: Using the previous part, prove that it is possible to divide any eight consecutive integers into two sets such that the sum of the integers in each set is the same, and the sum of the squares of the integers in each set is also the same.

Part iv: Show that given any eight consecutive integers one can split them into two sets of four such that the sum of the cubes in one set subtracted from the sum of the cubes in the second set is a constant, as in part ii.

Part v: Building on the pattern begun above, prove the following theorem by induction.

THEOREM: *Given an integer $k \geq 1$ it is possible to divide any 2^{k+1} consecutive integers into two sets $\{a_1, a_2, \ldots, a_{2^k}\}$ and $\{b_1, b_2, \ldots, b_{2^k}\}$ such that*

$$\begin{aligned} a_1 + a_2 + \cdots + a_{2^k} &= b_1 + b_2 + \cdots + b_{2^k} \\ a_1^2 + a_2^2 + \cdots + a_{2^k}^2 &= b_1^2 + b_2^2 + \cdots + b_{2^k}^2 \\ &\vdots \\ a_1^k + a_2^k + \cdots + a_{2^k}^k &= b_1^k + b_2^k + \cdots + b_{2^k}^k \end{aligned}$$

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A–Part i B–Part i: For both $n = 2$ and $n = 3$ a purely algebraic proof can be accomplished by expanding both sides, canceling common terms, and then grouping the remaining terms into a sum of squares. Omitting some intermediate algebra we find

$$\begin{aligned}(a_1^2 + a_2^2)(b_1^2 + b_2^2) &\geq (a_1b_1 + a_2b_2)^2 \\ \iff (a_1b_2 - a_2b_1)^2 &\geq 0\end{aligned}$$

and

$$\begin{aligned}(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) &\geq (a_1b_1 + a_2b_2 + a_3b_3)^2 \\ \iff (a_1b_2 - a_2b_1)^2 + (a_1b_3 - a_3b_1)^2 + (a_2b_3 - a_3b_2)^2 &\geq 0\end{aligned}$$

which proves the inequalities. Equality is achieved if and only if all summands on the left equal zero, which does imply that one sequence is a multiple of the other. For example, equality holds when $n = 2$ if and only if $a_1b_2 - a_2b_1 = 0$. If all the a_i and b_i are 0 then $(a_1, a_2) = \lambda(b_1, b_2)$ for any λ. Thus we assume that one of the variables is nonzero; without loss of generality $b_1 \neq 0$. Rearranging the above equation leads to $a_2 = \left(\frac{a_1}{b_1}\right) b_2$. Therefore defining $\lambda = \frac{a_1}{b_1}$ results in $(a_1, a_2) = \lambda(b_1, b_2)$. In either case we discovered that the sequences must be proportional to satisfy $a_1b_2 - a_2b_1 = 0$. This mini-result can now be applied to $n = 3$ to show fairly quickly that the two sequences are again proportional. Can the reader fill in the steps?

A more standard (and cleaner) proof involves vector algebra. Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$, then use the formulas $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$ (where θ is the angle between the two vectors), and $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + \cdots + a_nb_n$ to write two equal expressions for $(\mathbf{a} \cdot \mathbf{b})^2$. The inequality arises from the fact that $|\cos\theta| \leq 1$.

A–Parts ii,iii B–Parts ii,iii: This problem elicited a few astonishing proofs. Centreville High School managed to prove, using algebra and trigonometric identities, that the given statement was equivalent to an inequality in which a perfect square was greater than zero. Equally impressive was the paper submitted by Arcata High School in which the given inequality was proven equivalent to $\cos(\alpha - \beta) \leq 1$, which proves both parts ii and iii.

The proof we had in mind uses Cauchy-Schwarz with $n = 2$. Note that since α and β are in the first quadrant their sines and cosines are positive. We now figure out how to choose a_1, a_2, b_1, and b_2. The first factor in the product on the left hand side suggests that we let $a_1 = \sqrt{\cos^3\alpha/\cos\beta}$ and $a_2 = \sqrt{\sin^3\alpha/\sin\beta}$. With the second factor a course of action is less obvious. However, recalling that $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ we are led to assign $b_1 = \sqrt{\cos\alpha\cos\beta}$ and $b_2 = \sqrt{\sin\alpha\sin\beta}$. With these choices Cauchy-Schwarz implies

$$\begin{aligned}\left(\frac{\cos^3\alpha}{\cos\beta} + \frac{\sin^3\alpha}{\sin\beta}\right)(\cos(\alpha - \beta)) &\geq \left(\sqrt{\frac{\cos^3\alpha}{\cos\beta}}\sqrt{\cos\alpha\cos\beta} + \sqrt{\frac{\sin^3\alpha}{\sin\beta}}\sqrt{\sin\alpha\sin\beta}\right)^2 \\ &= (\cos^2\alpha + \sin^2\alpha)^2 = 1.\end{aligned}$$

It is crucial to understand in this next problem what needs to be proven. One has to show that the given equation implies that $\alpha = \beta$. Plugging in $\alpha = \beta$ and checking that the given equation works is doing just the reverse! Put another way, one is showing that $\alpha = \beta$ is one solution to the equation, but not showing that it is the *only* solution. A proof is obtained by noticing the similarity between the given equation and the inequality in the previous part. Since the inequality above was proven using Cauchy-Schwarz, equality is achieved if and only if the two sequences $\{a_1, a_2\}$ and $\{b_1, b_2\}$ are proportional, which is the same as requiring that $a_1 b_2 = a_2 b_1$. For our particular choices of a_1, a_2, b_1, and b_2 this becomes

$$\begin{aligned} \sqrt{\frac{\cos^3\alpha}{\cos\beta}}\sqrt{\sin\alpha\sin\beta} &= \sqrt{\frac{\sin^3\alpha}{\sin\beta}}\sqrt{\cos\alpha\cos\beta} \\ \iff \sin\beta\cos\alpha &= \sin\alpha\cos\beta \\ \iff \sin(\beta-\alpha) &= 0. \end{aligned}$$

However, $\beta - \alpha$ is an angle between $-90°$ and $90°$, and the only angle in this range whose sine is zero is the zero angle. Thus the last equation forces $\alpha = \beta$, W^5.

A–Part iv B–Part iv: For notational convenience denote $x_1^k + x_2^k + \cdots + x_n^k$ by S_k for k a positive integer. Cauchy-Schwarz immediately provides a useful inequality relating the S_k. Choosing our two sequences to be $\{x_1^{\frac{k+1}{2}}, x_2^{\frac{k+1}{2}}, \ldots, x_n^{\frac{k+1}{2}}\}$ and $\{x_1^{\frac{k-1}{2}}, x_2^{\frac{k-1}{2}}, \ldots, x_n^{\frac{k-1}{2}}\}$ we find that

$$\left(x_1^{k+1} + x_2^{k+1} + \cdots + x_n^{k+1}\right)\left(x_1^{k-1} + x_2^{k-1} + \cdots + x_n^{k-1}\right) \geq \left(x_1^k + x_2^k + \cdots + x_n^k\right)^2,$$

or $S_{k+1}S_{k-1} \geq S_k^2$ in shorthand notation. We now multiply a sequence of these inequalities together and cancel common terms (which are all positive) to obtain

$$\begin{aligned} S_{a+2}S_a &\geq S_{a+1}^2 \\ S_{a+3}S_{a+1} &\geq S_{a+2}^2 \\ &\vdots \\ S_b S_{b-2} &\geq S_{b-1}^2, \\ \Longrightarrow S_a S_b &\geq S_{a+1}S_{b-1}. \end{aligned}$$

The inequalities on both tests are of this type; choosing $a = 19$ and $b = 93$ yields the desired inequality for the A test while taking $a = 1$ and $b = 4$ finishes the B test.

An alternate solution involves expanding the products found on each side, canceling common terms, and finally showing that expressions like $x_1^a x_2^a (x_1^{b-a-1} - x_2^{b-a-1})(x_1 - x_2)$ are always zero or positive. We leave the details to the interested reader.

A–Part v B–Part v: One can often enhance the readability of a proof by using suggestive notation. So we let the k^{th} group contain p_k people, where $p_1 + p_2 + \cdots + p_m = n$ since there are a total of n people. In the same manner we let s_k be the side length of the k^{th} cake. Since no person may consume more than 25 cm^2 of cake and the area of the k^{th} piece of cake is s_k^2 we know that $s_k^2 \leq 25p_k$, or $s_k \leq 5\sqrt{p_k}$. The total amount of ribbon is merely $4s_1 + 4s_2 + \cdots + 4s_m$, which by the previous observation is less than or equal to $20\sqrt{p_1} + 20\sqrt{p_2} + \cdots + 20\sqrt{p_m}$. If we could only show that

$$20\sqrt{p_1} + 20\sqrt{p_2} + \cdots + 20\sqrt{p_m} \leq 20\sqrt{mn}$$

then we would be done. However, by Cauchy-Schwarz we know that

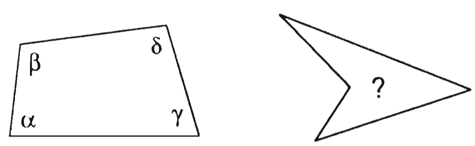
$$\begin{aligned} \left((\sqrt{p_1})^2 + (\sqrt{p_2})^2 + \cdots + (\sqrt{p_m})^2\right)\left(1^2 + 1^2 + \cdots + 1^2\right) &\geq (\sqrt{p_1} + \sqrt{p_2} + \cdots + \sqrt{p_m})^2 \\ \iff (p_1 + p_2 + \cdots + p_m)(m) &\geq (\sqrt{p_1} + \sqrt{p_2} + \cdots + \sqrt{p_m})^2 \\ \iff (n)(m) &\geq (\sqrt{p_1} + \sqrt{p_2} + \cdots + \sqrt{p_m})^2 \\ \iff \sqrt{mn} &\geq \sqrt{p_1} + \sqrt{p_2} + \cdots + \sqrt{p_m}, \end{aligned}$$

where the last step is reversible since both sides of the equation are positive. Multiplying both sides of the equation by 20 completes the proof.

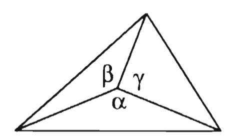
Divisions A and B Round Two Team Test December 1992

A–Parts i,ii B–Parts i,ii,iii: There are basically two possible configurations for the four points; either they form a convex quadrilateral or one point lies inside the triangle formed by the other three. These two cases are illustrated below. In the first case we know that the four angles marked α, β, δ, and γ sum to 360°. Therefore it is impossible for each

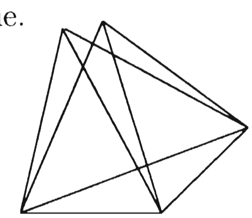
Case 1



Case 2



angle to be less than 90°. Since no angle is exactly 90° one of them must be obtuse, and it is clear how to form a triangle with three of the four points which includes this angle. (What would go wrong with this argument if the points did not form a convex quadrilateral, as shown in the figure with a question mark?) In the second case the three angles marked α, β, and γ sum to 360° and similar reasoning shows that one of these angles must be obtuse; it is again evident how to form a triangle which includes the obtuse angle.



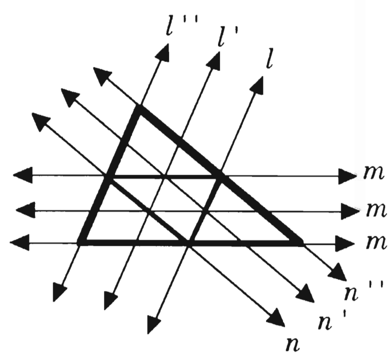
A popular approach to showing that at least a quarter of the triangles were obtuse went as follows. "Since we just proved that at least one of every four triangles is obtuse, and $\frac{1}{4}\binom{n}{3}$ is one-fourth of the total number of triangles, we are done." The problem with this reasoning is that it only works for four points at a time; if you try to generalize immediately to n points you may be overcounting the obtuse triangles. In the figure to the right two quadrilaterals are pictured. However, only one-seventh (not one-fourth) of the triangles shown are obtuse because the two quadrilaterals share the obtuse triangle! This is why the above argument is invalid.

To avoid overcounting we adopt another strategy. Given n points, let there be a total of k obtuse triangles. How many groups of four points contain a particular obtuse triangle? This is easy — the obtuse triangle already accounts for three of the four points, so there are $n-3$ points left which can have the honor of being the fourth point for a total of $n-3$ groups of four points. Since there are k obtuse triangles altogether at most $k(n-3)$ groups of four points contain one of them. (It is possible that there may be fewer since two of the triangles might be part of the same group of four points.)

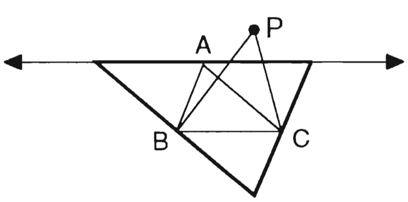
What would go wrong if $k < \frac{1}{4}\binom{n}{3}$? These k obtuse triangles are part of at most $k(n-3)$ groups of four points by the above. But

$$k(n-3) < \frac{1}{4}\binom{n}{3}(n-3) = \frac{n(n-1)(n-2)(n-3)}{4 \cdot 3 \cdot 2 \cdot 1} = \binom{n}{4}.$$

Since there are a total of $\binom{n}{4}$ groups of four points this means that there is a group of four points which doesn't contain any of the k obtuse triangles, contradicting our result from the first paragraph.



A–Parts iii,iv B–Part v: Let us make a few observations about medial triangles. It is simple to prove that in the first outer medial triangle all four smaller triangles are congruent. It follows, for example, that lines l and l' are parallel. Since the second outer medial triangle is formed from the first one in the same manner, we can further deduce that lines l, l', and l'' are parallel and equally spaced; as are lines m, m', and m'' and lines n, n', and n''. (The geometric details in a rigorous proof of this statement are tedious but should be obvious enough to omit from your write-up.)

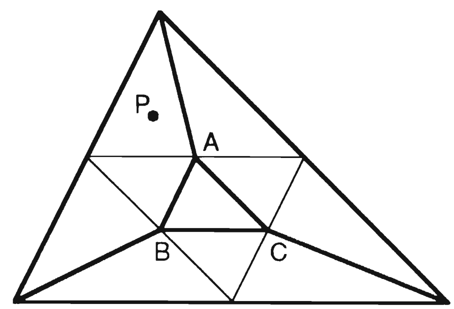


Choose three points A, B, and C which form a triangle of maximal area, which can certainly be done since there are only a finite number of triangles. If more than one triangle has the same maximal area any one of them will do. We claim that the first outer medial triangle of this triangle contains all the remaining points in its interior or on its sides. For if a point lay outside the first medial triangle then it must be located opposite one of the sides of the first outer medial triangle, such as the side containing A as indicated below. But if this is the case then $\triangle PBC$ has greater area than $\triangle ABC$ since $\triangle PBC$ has the same base but greater height than $\triangle ABC$. This contradicts the fact that $\triangle ABC$ was a triangle of maximal area. Therefore no points can lie outside the first outer medial triangle of points A, B, and C.

Before continuing, we pause to speculate on what other interesting problems are lurking within the diagram composed of nothing more than the plane and a collection of n points. What if we choose the three points which form the smallest or largest angle? What properties do the two points which are closest together have that possibly no other pair of points satisfy?

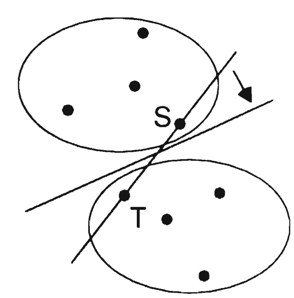
What about the two points farthest apart? What can be deduced about the three points which form the triangle of smallest perimeter? With a little imagination you can come up with some other "minimal" configurations and investigate the properties of those particular points. As incentive, try the following problem. Given n points in the plane, not all collinear, prove that one can find three points whose circumcircle contains none of the other points in its interior. All the ideas you need for a solution are listed somewhere in this paragraph.

It should come as no surprise to find that three points A, B, and C which form a triangle of minimal area also have interesting properties. We claim that no other point lies within the second outer medial triangle of $\triangle ABC$. If a point P does lie within the interior then it must be located in (or on the edges of) one of the four regions outlined by the heavy lines at right. Suppose that P lies inside $\triangle ABC$. Then clearly $\triangle PBC$ has smaller area, contradicting the way we chose A, B, and C. On the other hand, if P lies in one of the other regions as pictured above, then we can again form a triangle of smaller area. In this case P is closer to line AB than C is (by the observations above) so $\triangle PAB$ has smaller area than $\triangle ABC$, again providing a contradiction.



A–Part v B–Part iv: We first show that given any one of the $2n$ points (call it P) there exists a line through that point and another point in the set which divides the remaining $2n-2$ points in half. The proof of this fact may be faintly reminiscent of the geometric continuity proofs from last year. Construct a line through P and any other point (call it Q) in the set. If this line evenly divides the other $2n-2$ points then we are done. Otherwise one side of the line contains more than $n-1$ points and the other contains less. Now rotate the line 180° about point P. As it crosses each of the other points in turn the number of points on either side of the line increases or decreases by exactly one, since no three of the points are collinear. But when the line has rotated 180° back to Q the side which had greater than $n-1$ points has become the side which had less than $n-1$ points! Therefore at some time inbetween the line crossed a point R when the number of points on each side of the line was exactly balanced, so $\overleftrightarrow{PR}$ is our desired line. Now construct such a line through each of the $2n$ points. This process will not guarantee $2n$ distinct lines because some of the lines will be counted twice. However, since there are only two points on each line there will be at least n distinct lines, which was what we wanted.

It is not difficult to obtain a line dividing the $2n$ points in half, n on either side, once we have a line through two of the points, call them S and T, with $n-1$ points on either side. Simply rotate the latter line a small amount clockwise about the midpoint of segment ST, as demonstrated at right. (An amount small enough that our line doesn't cross over any other points, but only moves off of S and T.) We are not done, though, because we have to ensure that we still have n distinct divisions of our $2n$ points as outlined in the problem on the A division test. For example, is it possible to obtain the division shown at right by starting with a line through two points other than S and T and rotating a small amount clockwise? To see that the answer is no, perform the rotation in



reverse. It is clear that by rotating the division line at right in a counterclockwise direction, never crossing any of the points, we can only reach S and T, and no other pair of points. Therefore each of the n lines found in the previous paragraph yields a distinct division of our $2n$ points, completing the proof.

Divisions A and B — Round Three Team Test — January 1993

A–Parts i,ii B–Parts i,ii: Note that in these two problems a_1, a_2, a_3, and a_4 are ordinary real numbers which are determined by the four given equations. We don't yet know their exact values, but with a little patience one could solve the equations and determine them (as one school did). However, using the form of the given equations and a little ingenuity one can deduce some properties of the solutions without ever resorting to actually solving the equations.

Using the facts section it is straightforward to verify that

$$a_1(x+1)^3 + a_2(x+2)^3 + a_3(x+3)^3 + a_4(x+4)^3 \equiv (2x+1)^3.$$

Both polynomials are of degree three, so it suffices to find four values of x for which the two sides are equal. When $x = 0$ we need $a_1 + 8a_2 + 27a_3 + 64a_4 = 1$, which is true by the given equations. Similarly, the above equation is satisfied when $x = 1$, 2, and 3, using the other three equations defining the a_i. Since the cubics agree at four points they must be identically equal.

We let this polynomial equivalence do all the work for us. The coefficient of x^3 on the left hand side is clearly $a_1+a_2+a_3+a_4$, while the coefficient of x^3 on the right hand side is 8. Since identical polynomials have the same coefficients, it follows that $a_1 + a_2 + a_3 + a_4 = 8$. The second relationship follows from the fact that identical polynomials agree for every value of x. In particular, setting $x = -5$ in the above equation yields $-64a_1 - 27a_2 - 8a_3 - a_4 = -729$, which implies $64a_1 + 27a_2 + 8a_3 + a_4 = 729$ as desired.

A–Part iii B–Part iii: As suggested we consider the polynomial

$$f(x) = (x-1)(x-\frac{1}{2})\cdots(x-\frac{1}{n}).$$

If we write $f(x)$ in the form $f(x) = x^n+c_1x^{n-1}+\cdots+c_{n-1}x+c_n$ by multiplying out the factors we can obtain explicit expressions for the coefficients. We find that $c_1 = -(1+\frac{1}{2}+\ldots+\frac{1}{n})$, $c_2 = \frac{1}{1\cdot2}+\frac{1}{1\cdot3}+\cdots+\frac{1}{(n-1)\cdot n}$, $\ldots$, and $c_n = \frac{(-1)^n}{n!}$.

At this point we recognize these symmetric sums from the equation in the problem. Grouping similar powers of k and negating both sides, we can write the equation to be proved as

$$k(-\frac{1}{1}-\frac{1}{2}-\ldots-\frac{1}{n}) + k^2(\frac{1}{1\cdot2}+\frac{1}{1\cdot3}+\ldots+\frac{1}{(n-1)\cdot n}) + \cdots + k^n\frac{(-1)^n}{n!} = -1. \quad (*)$$

(In the B test solution substitute n for k.) Using the expressions for the c_i allows us to write equation $(*)$ as

$$c_1k + c_2k^2 + \cdots + c_nk^n = -1.$$

However, it is clear from the way that f was originally defined that $f(\frac{1}{k}) = 0$ for $k = 1, 2, \ldots, n$. In other words

$$\begin{aligned} f(\frac{1}{k}) = \left(\frac{1}{k}\right)^n + c_1\left(\frac{1}{k}\right)^{n-1} + \cdots + c_{n-1}\left(\frac{1}{k}\right) + c_n &= 0 \\ \iff 1 + c_1k + \cdots + c_{n-1}k^{n-1} + c_nk^n &= 0 \\ \iff c_1k + \cdots + c_{n-1}k^{n-1} + c_nk^n &= -1, \end{aligned}$$

which was what we needed to show.

A–Parts iv,v B–Parts iv,v: The form of the inequalities suggests using AM-GM. There is a product of quantities on the left which is less than an expression which includes an average of those same quantities. It is given that x and $r_1, r_2, \ldots,$ and r_n are all positive, so AM-GM is applicable to the n numbers $(x + r_1), (x + r_2), \ldots,$ and $(x + r_n)$ yielding

$$\begin{aligned} \sqrt[n]{(x + r_1)(x + r_2)\cdots(x + r_n)} &\leq \frac{(x + r_1) + (x + r_2) + \cdots + (x + r_n)}{n} \\ \Rightarrow \sqrt[n]{(x + r_1)(x + r_2)\cdots(x + r_n)} &\leq x + \frac{r_1 + r_2 + \cdots + r_n}{n} \\ \Rightarrow (x + r_1)(x + r_2)\cdots(x + r_n) &\leq \left(x + \frac{r_1 + r_2 + \cdots + r_n}{n}\right)^n, \end{aligned}$$

which was what we wanted (W^5). The B test solution reads precisely the same using $n = 4$.

The second inequality promises to be a little trickier since the product is now on the wrong side of the inequality sign for AM-GM to work directly. To gain some insight we attempt to prove the claim in the case $n = 4$ first. Expanding both sides yields

$$x^4 + x^3(r_1 + r_2 + r_3 + r_4) + \cdots \geq x^4 + x^3(4\sqrt[4]{r_1r_2r_3r_4}) + \cdots$$

It is immediately evident that the coefficient of x^3 on the left hand side is larger than the corresponding coefficient on the right hand side by AM-GM. This turns out to be true for every power of x. For example, the coefficient of x^2 on the left hand side is the sum of r_1r_2, r_1r_3, r_1r_4, r_2r_3, r_2r_4, and r_3r_4. Applying AM-GM to these six numbers yields

$$\begin{aligned} \frac{r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4}{6} &\geq \sqrt[6]{(r_1r_2r_3r_4)^3} \\ \Rightarrow r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4 &\geq 6\sqrt{r_1r_2r_3r_4}, \end{aligned}$$

which is exactly the coefficient of x^2 in the expansion of $(x + \sqrt[4]{r_1r_2r_3r_4})^4$. Hence term by term the left hand side of the original inequality is larger than the right, so we are done.

In general, by multiplying out the left hand side we see that the coefficient of each power of x is a sum to which we can apply AM-GM. The coefficient of x^{n-k} on the left hand side is just

$$(r_1r_2\ldots r_{k-1}r_k) + (r_1r_2\ldots r_{k-1}r_{k+1}) + \cdots + (r_{n-k+1}r_{n-k+2}\ldots r_n).$$

Notice that there are a total of $\binom{n}{k}$ terms in the above sum, and exactly $\binom{n-1}{k-1}$ of them contain r_1, because once we have designated r_1, there are only $n-1$ variables left from which to choose the $k-1$ remaining factors. The same reasoning applies to any of the r_i, so if we multiply all $\binom{n}{k}$ of the terms together we obtain $(r_1 r_2 \cdots r_n)^{\binom{n-1}{k-1}}$. Therefore applying AM-GM to those $\binom{n}{k}$ terms yields

$$\frac{(r_1r_2\ldots r_{k-1}r_k)+(r_1r_2\ldots r_{k-1}r_{k+1})+\cdots+(r_{n-k+1}r_{n-k+2}\ldots r_n)}{\binom{n}{k}} \geq (r_1r_2\ldots r_n)^{\frac{\binom{n-1}{k-1}}{\binom{n}{k}}}$$

$$\Rightarrow (r_1r_2\ldots r_{k-1}r_k)+(r_1r_2\ldots r_{k-1}r_{k+1})+\cdots+(r_{n-k+1}r_{n-k+2}\ldots r_n) \geq \binom{n}{k}(r_1r_2\ldots r_n)^{\frac{k}{n}}$$

$$\Rightarrow (r_1r_2\ldots r_{k-1}r_k)+(r_1r_2\ldots r_{k-1}r_{k+1})+\cdots+(r_{n-k+1}r_{n-k+2}\ldots r_n) \geq \binom{n}{n-k}(\sqrt[n]{r_1r_2\ldots r_n})^k.$$

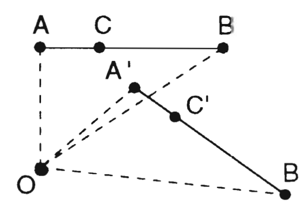
The verification that $\binom{n-1}{k-1}/\binom{n}{k} = \frac{k}{n}$ and $\binom{n}{k} = \binom{n}{n-k}$ are straightforward using the usual formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. But the right hand side of the last equation above is exactly the coefficient of x^{n-k} in the expansion of $(x+\sqrt[n]{r_1r_2\ldots r_n})^n$. We conclude that term by term the left hand side of the original inequality is larger than the right hand side, which proves the inequality.

Notice that in these proofs the argument began with inequalities guaranteed by AM-GM, which then implied the inequality sought after. A proof which begins with the statement to be proved and then works towards AM-GM is not necessarily correct!

Divisions A and B — Round Four Team Test — March 1993

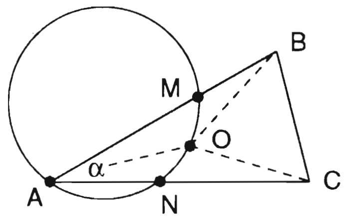
A–Parts i,ii B–Parts i,ii,iii: In several of the upcoming proofs we will develop intuitively obvious theorems from first principles, using only the distance preserving property of rotations and a few other elementary facts. We begin with a lemma.

LEMMA: If a rotation maps points A to A' and B to B' (as in the diagram) then it maps the segment AB to segment $A'B'$.



PROOF: A fundamental fact from geometry states that a point C is on segment AB if and only if $AC + CB = AB$. Let C' be the image of C in the above rotation. Since distances are preserved under rotation we know that $AC = A'C'$ and so on. Consequently $A'C' + C'B' = A'B'$ if and only if $AC + CB = AB$, or C' is on $\overline{A'B'}$ if and only if C is on $\overline{AB}$. Therefore all the points of $\overline{AB}$ rotate onto all of $\overline{A'B'}$.

The rest of the argument now flows smoothly. Label $\angle BAC = \alpha$. It is clear that a rotation about point A through angle α in the clockwise sense maps point A to itself and takes B to C, so by the lemma it takes $\overline{AB}$ to $\overline{AC}$. Also, if O is the circumcenter of $\triangle ABC$ then $OA = OB = OC$. Since $AB = AC$ we find by SSS that $\triangle AOB$ is congruent to $\triangle COA$; in particular $\angle BOA \cong \angle AOC$. Hence a rotation about O through this angle carries B to A and A to C, and consequently carries $\overline{BA}$ to $\overline{AC}$. It will be useful to compute this angle in terms of α. Since $\angle BAO \cong \angle CAO$ and $m\angle BAO + m\angle CAO = \alpha$ we find that both of these angles equal $\alpha/2$. Using $\triangle BAO$ it follows that $\angle BOA = 180° - \alpha$.



The rotation about O through an angle of $(180° - \alpha)$ takes segment BA to segment AC, and therefore maps M to some point on $\overline{AC}$. This point is the same distance from A as M is from B, as rotations preserve distance. There is only one such point, and N fits the bill, so N must be the point onto which M is rotated. We conclude that $OM = ON$ and $m\angle MON = 180 - \alpha$ by the definition of rotation. Finally we find that $m\angle MAN + m\angle NOM = \alpha + (180° - \alpha) = 180°$, so that A, M, O, and N all lie on the same circle.

A–Part iii: The strategy in most problems of this type is to find (or guess) a suitable point of common intersection and then show that all the circles pass through it. In this problem it is not too difficult to guess that the center of polygon $A_1A_2\ldots A_n$ is the proper point. (A regular polygon can be inscribed in a circle and we call the center O of this circle the center of the polygon.) Because this circumscribed circle passes through all the vertices the point O is the circumcenter of any triangle formed by three vertices of the polygon.

At this point the reader should retrieve his/her own diagram from amongst the stack of scratch paper generated while working on these questions. Now that polygon $B_1B_2\ldots B_n$ is neatly inscribed inside $A_1A_2\ldots A_n$ on your paper you will see that it seems reasonable for

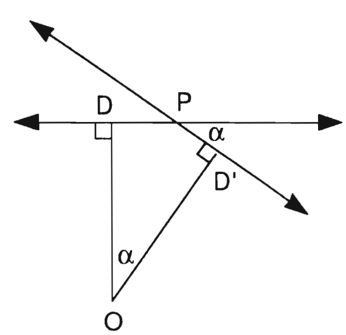
$$\triangle B_1A_2B_2 \cong \triangle B_2A_3B_3 \cong \cdots \cong \triangle B_nA_1B_1.$$

This isn't hard to show, for we can compute

$$\angle A_1B_1B_n = 180° - (\angle B_nB_1B_2 + A_2B_1B_2) = 180° - (B_1A_2B_2 + \angle A_2B_1B_2) = \angle A_2B_2B_1.$$

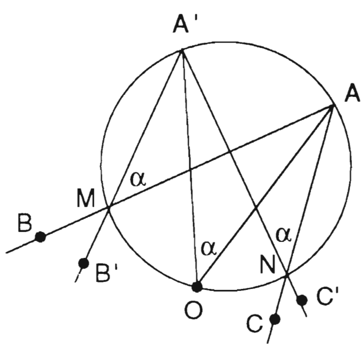
Similarly $\angle A_1B_nB_1 \cong \angle A_2B_1B_2$, so by ASA the two triangles are congruent, hence $A_1B_1 = A_2B_2$. We can now apply our previous result to isosceles triangle $A_1A_2A_3$ with points B_1 and B_2 on the sides to conclude that the circle through $\triangle B_1A_2B_2$ passes through O. By the same reasoning all the other circles pass through O as well, and we are done.

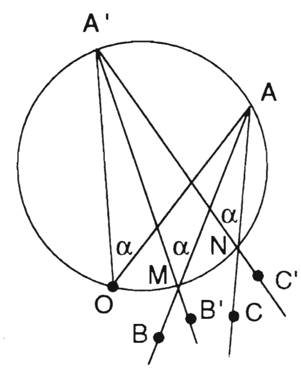
A–Part iv B–Part iv: Suppose that point O is actually on line l. Then line l', the image of line l under a rotation about O through angle α, must intersect l at point O. It is then clear that the angle between l and l' is α.



We now assume that O is not on line l. Let D be the unique point on l which is closest to O, which means that D is the foot of the perpendicular from O to l. When we rotate line l we obtain the new line l' and corresponding point D'. Since rotations preserve distances D' is the point on l' closest to O, which shows that OD' is perpendicular to l'. We can now easily calculate some angles. Let P be the point of intersection of l and l'. Using the fact that $\angle DOD' = \alpha$ (the angle of rotation) and $\angle ODP = \angle OD'P = 90°$ it follows that $\angle DPD' = 180° - \alpha$, so the angle of intersection between l and l' is α. Note that if l and l' were parallel then this would force $\alpha = 180°$, a case ruled out by the hypotheses of the question.

A–Part v B–Part v: Believe it or not the concluding question has one of the shortest proofs on the test. Let $\angle AOA' = \alpha$, the angle of rotation. The above result indicates that $\angle AMA' = \angle ANA' = \alpha$ as well, so by the facts section we conclude that both M and N lie on the circle through O, A, and A'. Notice that this argument works perfectly well whether O is inside, on, or outside the angle $\angle ABC$. Two of these cases are pictured below.





Of course we run into difficulties if the two rays forming the angle are rotated so far that they no longer intersect the original angle. However, if we rotate an intersecting pair of lines instead of an angle then the result will hold in all cases.

Divisions A and B — Round Five Team Test — April 1993

A–Part i: We choose to follow the advice given in the facts section to "work backwards." We begin by dividing both sides of equation (3) by two, yielding equation (1). Now that we have derived $a+b+c = d+e+f$ we cube this equation and subtract three times equation (4), which yields precisely equation (2) since

$$(a+b+c)^3 = a^3 + b^3 + c^3 + 3(a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2 + 2abc), \text{ and}$$
$$3(a+b)(b+c)(c+a) = 3(a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2 + 2abc).$$

Thus equations (3) and (4) together imply equations (1) and (2).

A–Part ii B–Part i: As mentioned in the facts section, equivalent systems of equations have the same solutions. Thus to find solutions to (1) and (2) we instead concentrate on equations (3) and (4) since they turn out to be easier to deal with. We can further simplify matters by substituting $u = a+b$, $v = a+c$, $\ldots$, $z = e+f$ so that our equations become

$$u + v + w = x + y + z \quad \text{and} \quad uvw = xyz. \tag{$*$}$$

Once we find integer solutions to these equations we can recover a through f by noting that $a = \frac{1}{2}(u + w - v)$, $b = \frac{1}{2}(u + v - w)$, $c = \frac{1}{2}(v + w - u)$, and so on. As long as the sum $u + v + w$ is even each expression like $(u + w - v)$ will also be even so that the six numbers $a, b, \ldots, f$ will in fact be integers.

It should be easy to find solutions to $(*)$ since there are six variables and only two equations, but the restriction that they be integers means that we still have to be careful. To begin, we let u, v, x, and y be fixed and treat only w and z as variables. We have two linear equations in these variables which can be easily be solved, yielding

$$z = \left(\frac{uv}{xy - uv}\right)(x + y - u - v) \quad \text{and} \quad w = \left(\frac{xy}{uv}\right) z.$$

We now make that observation that our life would be much easier if we choose u, v, x, and y so that $xy = 2uv$, because then the above equations would simplify to

$$z = x + y - u - v \quad \text{and} \quad w = 2z,$$

and we no longer have to worry about fractions. It now remains to choose some values of u, v, x, and y and see what happens. We found that choosing $u = 6$, $v = 10$, $x = 15$, and $y = 8$ (so that $xy = 2uv$) leads to $z = 7$ and $w = 14$. Recovering a through f as indicated above yields $a = 5$, $b = 1$, $c = 9$, and $d = 7$, $e = 8$, $f = 0$. Sure enough, these two sets of three numbers have the same sum (15) and the same sum of cubes (855) so they are a solution to equations (1) and (2).

There are a surprising number of solutions to the original equations (1) and (2), some of them clever, some of them quite nonobvious, and all within your reach now. For example, $a = 1$, $b = 1$, $c = 1$, and $d = 4$, $e = 4$, $f = -5$ is a well-known solution. Try your hand at discovering a few more.

B–Part ii: This is a short exercise in algebra which will help with the next part. We let our four consecutive squares be $(x+1)^2$, $(x+2)^2$, $(x+3)^2$, and $(x+4)^2$. Performing the indicated operations we find

$$\begin{aligned}(x+1)^2 + (x+4)^2 - (x+2)^2 - (x+3)^2 &= (x^2 + 2x + 1) + (x^2 + 8x + 16) - \\ &\quad -(x^2 + 4x + 4) - (x^2 + 6x + 9) \\ &= 4,\end{aligned}$$

which is a constant independent of which four consecutive squares we chose.

A–Part iii, B–Part iii: The previous part motivates the following idea. Label our consecutive integers $x + 1$ through $x + 8$. Split the first four integers into two sets of two as described above, and then do the same with the second four integers, yielding the sets $\{x+1, x+4\}$, $\{x+2, x+3\}$ and $\{x+5, x+8\}$, $\{x+6, x+7\}$. In each case the sum of the squares in one set is four greater than the sum of the squares in the other set. Thus pair the larger group from the first four integers with the smaller group from the second four integers! This pairing produces the two sets $\{x+1, x+4, x+6, x+7\}$ and $\{x+2, x+3, x+5, x+8\}$ and ensures that both the plain sum and the sum of squares within each set will be equal. Of course, one could also perform the necessary algebra to verify that this manner of splitting $x+1$ through $x+8$ does the job. (Try it.)

A–Part iv, B–Part iv: Our intuition says that the obvious candidate for a method of splitting up the eight integers is to try our answer from the previous part. This does indeed work, as is immediately verified by checking that

$$(x+2)^3+(x+3)^3+(x+5)^3+(x+8)^3-(x+1)^3-(x+4)^3-(x+6)^3-(x+7)^3=48,$$

which is independent of x. (We omitted a little algebra.) However, it is important to understand why this works. Expand each of the above cubes according to the formula $(x+a)^3 = x^3+3ax^2+3a^2x+a^3$. Clearly the x^3 terms cancel. The coefficient of the x^2 term is $3(2+3+5+8-1-4-6-7)$ which equals zero by construction — we showed in the previous part that the sets of numbers $\{2,3,5,8\}$ and $\{1,4,6,7\}$ have the same sum. We also showed that they have the same sum of squares, so the coefficient of the x term, which is $3(2^2+3^2+5^2+8^2-1^2-4^2-6^2-7^2)$ must also equal zero. This leaves us with only a constant term, as desired.

A–Part v, B–Part v: We have all the ingredients necessary to prove the main theorem at this point; we just apply the above ideas repeatedly by using an induction. The base case $k=1$ is trivial and we did the case $k=2$ above. Now assume that we have found a way to divide any 2^k consecutive integers into two sets so that the sums of their first powers, squares, all the way up to their $(k-1)^{\text{st}}$ powers are equal. This is the induction hypothesis. The sums of their k^{th} powers won't be equal; but, just as above, the difference between these two sums of k^{th} powers will be a constant, since the coefficients of x, x^2, ..., and x^k will all vanish by the induction hypothesis, just as they did in part iv.

Now suppose we are given any 2^{k+1} consecutive integers. Consider the first 2^k integers and the second 2^k integers separately. By the induction hypothesis we can split each group of 2^k integers into two sets so that the sums of the first through $(k-1)^{\text{st}}$ powers are equal. Also, by the above argument the sums of k^{th} powers differ by a constant. Thus pair the set with the smaller sum of k^{th} powers from the first 2^k integers with the set with the larger sum of k^{th} powers from the second 2^k integers, and then pair the remaining two sets together. We have now divided all 2^{k+1} integers into two sets so that not only are the sums of the first through $(k-1)^{\text{st}}$ powers equal, but also the sum of the k^{th} powers, just as in part iii. This completes the induction step, and we're done.

1993-1994

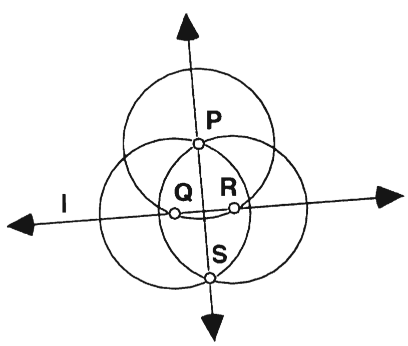
The Fourth Year of the Mandelbrot Competition

Mandelbrot Morsels

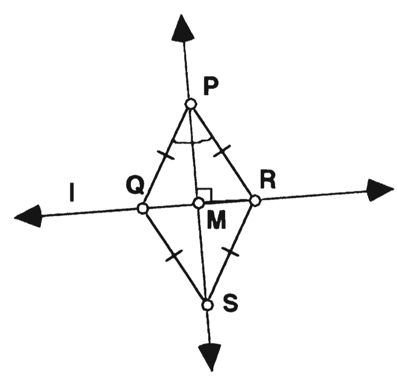
An Introduction To Construction

1993-94

The elementary theorems concerning congruent triangles or arcs of circles create a neat logical foundation for plane geometry and lead to such beautiful results as Ptolemy's theorem or the nine point circle. However, it is also satisfying to be able to use them in a more concrete manner. Therefore we will turn out attention to the realm of "applied geometry," or constructions. By construction we mean any operation involving only a straightedge and compass. Here are the rules for their use. A straightedge may be used to create a line, segment, or ray on two points, and may also be used to create an arbitrary line passing through one point. A compass may be opened to a width equal to the distance between a given pair of points in the diagram and then placed with its center at any point in the plane to create a circle with the chosen center and radius. One is also allowed to designate an arbitrary point on any geometric object such as a line, circle, ray, segment, and so on.



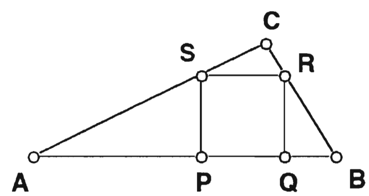
The typical construction problem invariably follows this pattern: given a collection of geometric objects in the plane construct another geometric object which has a specific property. Here is a typical construction problem, "Given a line l and a point P not on this line construct a line passing through point P which is perpendicular to l." Here is the solution to this construction problem; perhaps you already know how it goes. Choose an arbitrary point Q on line l. Draw the circle with center P and radius PQ; this circle intersects l in a second point besides Q, call it R. Keeping the compass open to the same radius PQ draw two more circles centered at Q and R. These two circles intersect at two points; one of them is P; label the other point S. Then line $\overleftrightarrow{PS}$ is the desired line.



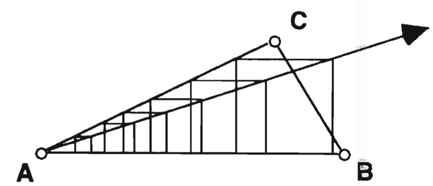
Naturally you, the discerning reader, are not convinced by this fancy manipulation of straightedge and compass. "I don't buy it," you say. Indeed, the description of the construction is only half of the solution. Therefore I shall prove that $\overleftrightarrow{PS}$ is the desired line. By construction we have $PQ = PR = QS = RS$, as they are all congruent radii. Therefore we may deduce that $\triangle PQS \cong \triangle PRS$ by the side-side-side criteria for congruent triangles, and hence $\angle QPS \cong \angle RPS$. Let the intersection of lines $\overleftrightarrow{PS}$ and l be M. By the side-angle-side criteria for congruent triangles we conclude that $\triangle QPM \cong \triangle RPM$, thus $\angle PMQ \cong \angle PMR$. But these two angles sum to 180°, so each must be a right angle. This argument shows that line $\overleftrightarrow{PS}$ is perpendicular to line l, as we claimed.

There are a few characteristics of the above style of proof that are worth highlighting. First, the solution consisted of two parts: a description of the construction and a proof of its validity. Each part will be worth half credit on a team test. In addition, all geometric objects were labeled and a diagram was included. This makes the proof easy to follow and unambiguous. Finally, note that only one solution was presented. This does not mean there could not be more of them! In this example it turns out that $\overleftrightarrow{PM}$ is the only possible line that passes through P and is perpendicular to l. Can you see why?

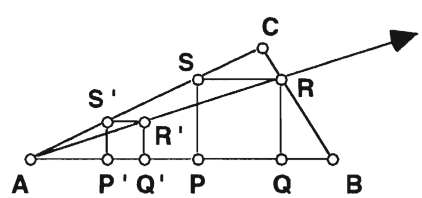
There are a host of other simple constructions like the above example which one frequently wishes to use as steps in a more complicated construction. Here are the most commonly used ones: creating a line through a given point that is parallel or perpendicular to another given line, bisecting a segment or an angle, copying a segment or an angle, and constructing an equilateral triangle (or 60° angle). It would be highly educational (hint, hint) to pull out pencil and paper or a geometry textbook in order to figure out how one might accomplish a few of the above constructions if they are unfamiliar.



On to a more interesting problem. Given $\triangle ABC$ with $\angle C$ obtuse, construct an inscribed square — that is, find points P, Q, R, and S such that P and Q lie on $\overline{AB}$, R lies on $\overline{BC}$, S lies on $\overline{AC}$, and $PQRS$ is a square, as shown in the diagram to the left.



To solve this problem we employ the clever method of *relaxing a constraint.* In our problem this means that for the time being we won't worry about requiring point R to lie on segment $\overline{BC}$. This has the advantage of making the problem much simpler. We can construct a square satisfying the rest of the conditions by choosing any point on $\overline{AC}$, dropping a perpendicular segment to $\overline{AB}$, and then building a square with this segment as its left hand side. Imagine all the possible squares that can be constructed in this manner, ranging from tiny squares near vertex A to large ones that extend outside $\triangle ABC$. For one of these squares the upper right hand vertex will lie on $\overline{BC}$ — that is the square we are interested in. To figure out how to obtain that one, consider the collection of all points which are upper right hand corners of a partially inscribed square. It is a ray with one endpoint at A, as shown in the diagram. Finding the "correct square" is now straightforward; just choose the point where this ray intersects $\overline{BC}$ and use that point as the upper right vertex.



This idea yields the following construction. Choose an arbitrary point S' on $\overline{AC}$, drop a perpendicular to P' on $\overline{AB}$, and construct the square $P'Q'R'S'$ with left hand side $\overline{P'S'}$. Draw ray $\overrightarrow{AR'}$ and let R be the point where this ray intersects $\overline{BC}$. Now drop a perpendicular from R to Q on $\overline{AB}$ and also construct a parallel to $\overline{AB}$ through R which intersects $\overline{BC}$ at S. Finally let P be the foot of the perpendicular from S to $\overline{AB}$. Essentially we just inscribed rectangle $PQRS$ inside triangle ABC. If we can show that this rectangle is a square then we would be done. However, this is not difficult to prove; essentially one uses all the similar triangles in the diagram (such as $\triangle AP'S' \sim \triangle APS$) to show that figures $P'Q'R'S'$ and $PQRS$ are similar. Since $P'Q'R'S'$ is a square (by construction), then $PQRS$ must be one also. And you're done!

The Mandelbrot Competition

Division A Round One Team Test

Definitions: An elementary concept in probability is that of *expected value.* Suppose that some experiment can yield k possible values, each of them equally likely. Call these possible values $a_1, a_2, \ldots, a_k$. We define the expected value to be $(a_1 + a_2 + \cdots + a_k)/k$. For instance, if the experiment were rolling a die then the possible outcomes are 1, 2, 3, 4, 5, or 6. Each possibility occurs with equal probability so the expected roll is $(1+2+3+4+5+6)/6 = 3\frac{1}{2}$.

Setup: Richard owns n cars. The first has a top speed of 10 mph, the next travels at up to 20 mph, and so on up to the n^{th}, which can move as fast as $10n$ mph. The cars are initially lined up in a random order on a one lane road. When they begin to move clumps will form as faster cars are held up by slower cars in front of them. More precisely, a block of m cars in a row form a *cluster* if the lead car in that block is the slowest of the m cars, if the lead car is faster than the car immediately behind the cluster, and if the lead car is slower than all the cars in front of it. For example, if the cars are lined up so their speeds are

$$30 \quad 10 \,|\, 70 \quad 20 \,|\, 40 \,|\, 60 \quad 50 \quad \longrightarrow$$

in that order and they start traveling to the right then four clusters will form as indicated by the vertical divisions.

In general there are $n!$ ways to order the n cars; each arrangement occurs with equal probability and in each arrangement a certain number of clusters form. Let C_n be the sum of these numbers, so that C_n is the total number of clusters which form in all $n!$ orderings. The goal of this team test will be to show that the expected number of clusters is $1 + \frac{1}{2} + \cdots + \frac{1}{n}$.

Problems:

Part i: Compute the expected number of clusters to form for $n = 1$, 2, and 3. Compare these values to those predicted by the formula given above.

Part ii: Show that

$$C_{n+1} = (n+1)C_n + n!.$$

Hint: Think of the $(n+1)!$ orderings of $n+1$ cars as orderings of the first n cars with the fastest car (the one moving at $10(n+1)$ mph) inserted at different positions.

Part iii: Show that if we define $A_n = n!\,(1 + \frac{1}{2} + \cdots + \frac{1}{n})$ for $n \geq 1$ then the A_n satisfy the recursion $A_{n+1} = (n+1)A_n + n!$.

Part iv: Finish the proof by showing that the expected number of clusters to form with n cars is $1 + \frac{1}{2} + \cdots + \frac{1}{n}$.

Part v: Bonus question. Now that you have completed the above proof, compute the expected *size* of a cluster picked at random from among all the clusters that form in all the $n!$ orderings of n cars.

November 1993

The Mandelbrot Competition

Division B Round One Team Test

Definitions: An elementary concept in probability is that of *expected value.* Suppose that some experiment can yield k possible values, each of them equally likely. Call these possible values $a_1, a_2, \ldots, a_k$. We define the expected value to be $(a_1 + a_2 + \cdots + a_k)/k$. For instance, if the experiment were rolling a die then the possible outcomes are 1, 2, 3, 4, 5, or 6. Each possibility occurs with equal probability so the expected roll is $(1+2+3+4+5+6)/6 = 3\frac{1}{2}$.

Setup: Richard owns n cars. The first has a top speed of 10 mph, the next travels at up to 20 mph, and so on up to the n^{th}, which can move as fast as $10n$ mph. The cars are initially lined up in a random order on a one lane road. When they begin to move clumps will form as faster cars are held up by slower cars in front of them. More precisely, a block of m cars in a row form a *cluster* if the lead car in that block is the slowest of the m cars, if the lead car is faster than the car immediately behind the cluster, and if the lead car is slower than all the cars in front of it. For example, if the cars are lined up so their speeds are

$$30 \quad 10 \,|\, 70 \quad 20 \,|\, 40 \,|\, 60 \quad 50 \quad \longrightarrow$$

in that order and they start traveling to the right then four clusters will form as shown.

In general there are $n!$ ways to order the n cars; each arrangement occurs with equal probability and in each arrangement a certain number of clusters form. Let C_n be the sum of these numbers, so that C_n is the total number of clusters which form in all $n!$ orderings. The goal of this team test will be to show that the expected number of clusters is $1 + \frac{1}{2} + \cdots + \frac{1}{n}$.

Problems:

Part i: Show that $C_1 = 1$, $C_2 = 3$, and $C_3 = 11$. Use these figures to calculate the expected number of clusters that form for $n = 1$, 2, and 3.

Part ii: Suppose that a particular ordering for n cars is given which forms k clusters. Show that if a car faster than all of the first n cars is added at any spot in the line, then the new arrangement also forms k clusters unless the new car is put at the front of the line, in which case $k + 1$ clusters are formed.

Part iii: Using the idea from the previous step show that

$$C_{n+1} = (n+1)C_n + n!.$$

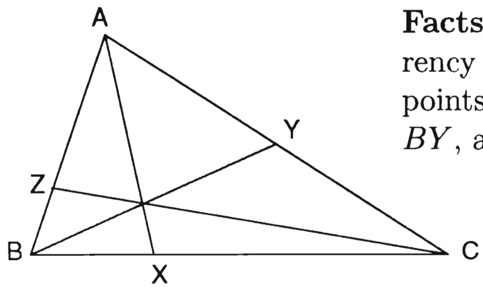
Part iv: Show that if we define $A_n = n!\,(1 + \frac{1}{2} + \cdots + \frac{1}{n})$ for $n \geq 1$ then the A_n satisfy the recursion $A_{n+1} = (n+1)A_n + n!$.

Part v: Show that $A_n = C_n$ for $n = 1$, 2, and 3. Argue that $A_n = C_n$ for all $n \geq 1$. Now finish the proof by showing that $1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is the expected number of clusters.

December 1993

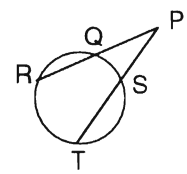
The Mandelbrot Competition

Division A Round Two Team Test



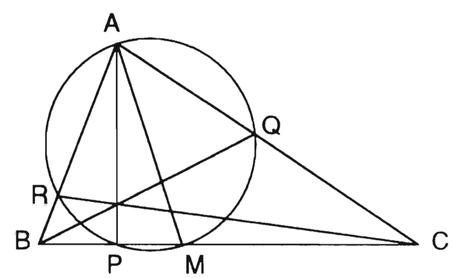
Facts: *Ceva's theorem* is a powerful tool for proving concurrency of lines in a triangle. It states that if X, Y, and Z are points on segments BC, AC, and AB of $\triangle ABC$ then AX, BY, and CZ are concurrent if and only if

$$\frac{(AZ)(BX)(CY)}{(AY)(BZ)(CX)} = 1.$$



One of the most useful theorems of plane geometry is the *Power of a Point* theorem, which says that $(PQ)(PR) = (PS)(PT)$, where the points are situated as shown to the right. The theorem also holds if P is inside the circle; in this case P is a point on the intersecting chords $\overline{QR}$ and $\overline{ST}$.

Setup: Given a triangle $\triangle ABC$ let M be the foot of the angle bisector from vertex A to side BC. Construct the circle with AM as diameter; this circle intersects lines AB and AC in exactly one other point besides A, call these points R and Q respectively. Similarly this circle usually intersects line BC in one other point besides point M, call this point P. If line BC happens to be tangent to the circle at M we shall say that point P is the same as point M. The goal of this test will be to prove that AP, BQ, and CR are concurrent for any triangle $\triangle ABC$.



Problems:

Part i: You will need to consider three cases. First prove that if $\angle B$ is a right angle then AP, BQ, and CR are concurrent at point B. Next show that if $\angle B$ is obtuse then points P and R are located outside the triangle. (Similar considerations apply to $\angle C$.) Finally, show that if both $\angle B$ and $\angle C$ are acute then points P, Q, and R do lie on the sides of the triangle, as shown in the setup diagram above.

Part ii: Show that $\overline{AQ} \cong \overline{AR}$ in all cases.

Part iii: We will attack the case where both $\angle B$ and $\angle C$ are acute. Prove that

$$\frac{(AB)(CM)}{(BM)(CA)} = \frac{(BP)(CQ)}{(BR)(CP)}.$$

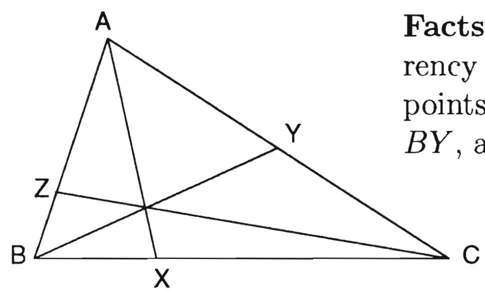
Part iv: Complete the acute case by proving that AP, BQ, and CR are concurrent.

Part v: Finish this proof by using similar reasoning as outlined in parts ii, iii, and iv to prove the assertion in the case where either $\angle B$ or $\angle C$ is obtuse.

December 1993

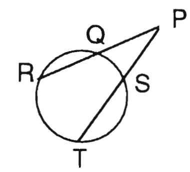
The Mandelbrot Competition

Division B Round Two Team Test

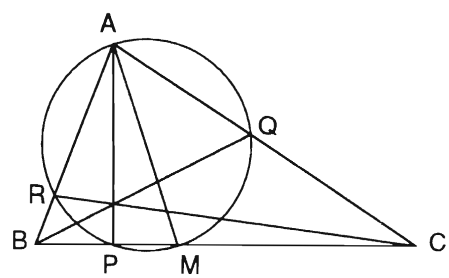


Facts: *Ceva's theorem* is a powerful tool for proving concurrency of lines in a triangle. It states that if X, Y, and Z are points on segments BC, AC, and AB of $\triangle ABC$ then AX, BY, and CZ are concurrent if and only if

$$\frac{(AZ)(BX)(CY)}{(AY)(BZ)(CX)} = 1.$$



One of the most useful theorems of plane geometry is the *Power of a Point* theorem, which says that $(PQ)(PR) = (PS)(PT)$, where the points are situated as shown to the right. The theorem also holds if P is inside the circle; in this case P is a point on the intersecting chords $\overline{QR}$ and $\overline{ST}$.



Setup: Given an acute triangle $\triangle ABC$ let M be the foot of the angle bisector from vertex A to side BC. Construct the circle with AM as diameter; this circle intersects lines AB and AC in exactly one other point besides A, call these points R and Q respectively. Similarly this circle usually intersects line BC in one other point besides point M, call this point P. If line BC happens to be tangent to the circle at M we shall say that point P is the same as point M. The goal of this test will be to prove that AP, BQ, and CR are concurrent.

Problems:

Part i: Show that $\overline{AQ} \cong \overline{AR}$.

Part ii: Establish the following lemma. If $\triangle ABC$ is an acute triangle then the foot of the altitude from A to line BC lies between points B and C.

Part iii: Use the lemma from part ii to prove that points P, Q, and R actually lie on the sides of the triangle as shown above, since $\triangle ABC$ is an acute triangle.

Part iv: Use the Power of a Point theorem to show that

$$\frac{(AB)(CM)}{(BM)(CA)} = \frac{(BP)(CQ)}{(BR)(CP)}.$$

Part v: Now combine parts i and iv, the Angle Bisector theorem, and Ceva's theorem to prove that AP, BQ, and CR are concurrent.

January 1994

The Mandelbrot Competition

Division A Round Three Team Test

Definitions: We say that a function $f(x,y)$ of two variables is symmetric if $f(x,y) = f(y,x)$ for all possible x and y. If one is given an explicit formula for $f(x,y)$, then it is sufficient to check that replacing y by x and x by y does not change the value of the formula. For example, consider $p(x,y) = x^2 + 3xy + y^2$ and $q(x,y) = \sin(x-y)$. We find that $p(y,x) = y^2 + 3yx + x^2 = p(x,y)$, so $p(x,y)$ is symmetric. However $q(y,x) = \sin(y-x) = -\sin(x-y) \neq q(x,y)$; hence $q(x,y)$ is not symmetric.

This concept can be easily extended to a function of n variables. We say $f(x_1, \ldots, x_n)$ is symmetric in n variables if interchanging any two of the variables does not affect the value of the function.

Setup: We create three sequences of polynomials as follows. Define

$$f_1(x,y) = x, \quad g_1(x,y) = y, \quad \text{and} \quad h_1(x,y) = f_1(x,y) + g_1(x,y) = x + y.$$

Now define the rest of each sequence recursively by setting

$$f_{n+1}(x,y) = f_n(x,y) \cdot h_n(x,y), \quad g_{n+1}(x,y) = g_n(x,y) \cdot f_n(y,x), \quad \text{and}$$
$$h_{n+1}(x,y) = f_{n+1}(x,y) + g_{n+1}(x,y).$$

Notice that the second factor in the definition of $g_{n+1}(x,y)$ is $f_n(y,x)$, not $f_n(x,y)$! The goal of this team test will be to prove that $h_n(x,y)$ is symmetric for all positive integers n.

Problems:

Part i: Let $p(x,y)$ be a function whose domain and range are the real numbers. Prove or find a counterexample to the following assertion: "If $[p(x,y)]^3$ is a symmetric function, then $p(x,y)$ is also a symmetric function."

Part ii: Suppose that $x + y + z = \frac{\pi}{2}$. Prove that in this case the function

$$q(x,y,z) = \frac{\sin x \cos x}{\frac{\cos y}{\sin y} + \frac{\cos z}{\sin z}}$$

is symmetric in the variables x, y, and z whenever it is defined.

Part iii: Compute f_n, g_n, and h_n for $n = 1$, 2, and 3. Verify that in each case h_n is symmetric.

Part iv: Show that $h_n(x,y)$ is symmetric if

$$f_n(x,y) - f_n(y,x) = g_n(y,x) - g_n(x,y). \qquad (*)$$

Part v: Prove $(*)$ by induction, thus completing the proof that h_n is symmetric for all positive integers n.

January 1994

The Mandelbrot Competition

Division B Round Three Team Test

Definitions: We say that a function $f(x, y)$ of two variables is symmetric if $f(x, y) = f(y, x)$ for all possible x and y. If one is given an explicit formula for $f(x, y)$, then it is sufficient to check that replacing y by x and x by y does not change the value of the formula. For example, consider $p(x, y) = x^2 + 3xy + y^2$ and $q(x, y) = \sin(x - y)$. We find that $p(y, x) = y^2 + 3yx + x^2 = p(x, y)$, so $p(x, y)$ is symmetric. However $q(y, x) = \sin(y - x) = -\sin(x - y) \neq q(x, y)$; hence $q(x, y)$ is not symmetric.

Setup: We create three sequences of polynomials as follows. Define

$$f_1(x, y) = x, \quad g_1(x, y) = y, \quad \text{and} \quad h_1(x, y) = f_1(x, y) + g_1(x, y) = x + y.$$

Now define the rest of each sequence recursively by setting

$$f_{n+1}(x, y) = f_n(x, y) \cdot h_n(x, y), \quad g_{n+1}(x, y) = g_n(x, y) \cdot f_n(y, x), \quad \text{and}$$
$$h_{n+1}(x, y) = f_{n+1}(x, y) + g_{n+1}(x, y).$$

Notice that the second factor in the definition of $g_{n+1}(x, y)$ is $f_n(y, x)$, not $f_n(x, y)$! The goal of this team test will be to prove that $h_n(x, y)$ is symmetric for all positive integers n.

Problems:

Part i: Let $p(x, y)$ be a function whose domain and range are the real numbers. Prove or find a counterexample to the following assertion: "If $[p(x, y)]^2$ is a symmetric function, then $p(x, y)$ is also a symmetric function."

Part ii: Compute f_n, g_n, and h_n for $n = 1$, 2, and 3. Verify that in each case h_n is symmetric.

Part iii: Write down the equation which "says" that $h_n(x, y)$ is symmetric. Use the definition of h_n above to show that $h_n(x, y)$ is symmetric if

$$f_n(x, y) - f_n(y, x) = g_n(y, x) - g_n(x, y). \qquad (*)$$

Part iv: Check that $(*)$ is true for $n = 1$ and 2. Also, rewrite the recursive definitions for f_{n+1} and g_{n+1} solely in terms of f_n and g_n, i.e. eliminate h_n from the recursions.

Part v: Using your preparations in part iv prove $(*)$ by induction, thus completing the proof that h_n is symmetric for all positive integers n.

March 1994

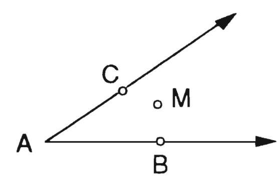
The Mandelbrot Competition

Division A Round Four Team Test

Facts: A *half-turn* about a point O in the plane is a geometric transformation which rotates every point 180° about the point O. Imagine placing a tack in the plane at the point O and then spinning the entire plane halfway around, keeping the point O fixed. It follows that if a point A gets mapped to the point A' by a half-turn then O is the midpoint of $\overline{AA'}$.

Here is a short list of basic constructions: creating a line through a given point that is parallel or perpendicular to another given line, bisecting a segment or an angle, copying a segment or an angle, drawing a circle given its center and radius, and constructing an equilateral triangle. You may use any of these instructions in the description of a major construction without detailing exactly how they are performed.

Setup: Most of the problems below involve the diagram to the right. You are given an angle $\angle BAC$ which can be acute, right, or obtuse. You are also given a point M in the interior of the angle.



Problems:

Part i: Construct a segment $\overline{PQ}$ with P on $\overrightarrow{AB}$ and Q on $\overrightarrow{AC}$ such that M is the midpoint of $\overline{PQ}$. Since midpoints are involved an appropriate half turn is very useful.

Part ii: Any line passing through point M that intersects $\overrightarrow{AB}$ and $\overrightarrow{AC}$ creates a triangle. Prove that of all such triangles the one with the smallest area is the one you constructed in part i. Perform the actual construction of this triangle and include it with your proof.

Part iii: There are two circles lying in the interior of $\angle BAC$ which are tangent to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and pass through point M. One is smaller and closer to point A than the other. We shall call the closer one the *inner circle* and the further one the *outer circle.* Describe how to construct the outer circle.

Part iv: Construct a segment $\overline{PQ}$ with P on $\overrightarrow{AB}$ and Q on $\overrightarrow{AC}$ such that $\overline{PQ}$ passes through point M and $AP + PM = AQ + QM$. The outer circle will be a step in your construction.

Part v: Construct a segment $\overline{PQ}$ with P on $\overrightarrow{AB}$ and Q on $\overrightarrow{AC}$ passing through M such that $AP - PM = AQ - QM$. You need not perform the actual constructions for parts iv and v, just describe the steps and prove that they yield the segment $\overline{PQ}$ with the desired properties.

March 1994

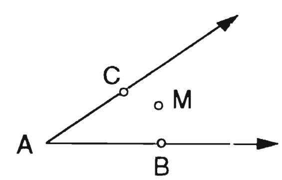
The Mandelbrot Competition

Division B Round Four Team Test

Facts: A *half-turn* about a point O in the plane is a geometric transformation which rotates every point 180° about the point O. Imagine placing a tack in the plane at the point O and then spinning the entire plane halfway around, keeping the point O fixed. It follows that if a point A gets mapped to the point A' by a half-turn then O is the midpoint of $\overline{AA'}$.

Here is a short list of basic constructions: creating a line through a given point that is parallel or perpendicular to another given line, bisecting a segment or an angle, copying a segment or an angle, drawing a circle given its center and radius, and constructing an equilateral triangle. You may use any of these instructions in the description of a major construction without detailing exactly how they are performed.

Setup: Most of the problems below involve the diagram to the right. You are given an angle $\angle BAC$ which can be acute, right, or obtuse. You are also given a point M in the interior of the angle.



Problems:

Part i: Given a point M and a line $\overleftrightarrow{AB}$, describe how to construct the line which is the image of $\overleftrightarrow{AB}$ under a half turn about M.

Part ii: Construct a segment $\overline{PQ}$ with P on $\overrightarrow{AB}$ and Q on $\overrightarrow{AC}$ such that M is the midpoint of $\overline{PQ}$. The previous problem is part of the construction. Perform the actual construction and include it with your proof.

Part iii: There are two circles lying in the interior of $\angle BAC$ which are tangent to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and pass through point M. One is smaller and closer to point A than the other. We shall call the closer one the *inner circle* and the further one the *outer circle*. Describe how to construct the outer circle.

Part iv: Let l be the line which is tangent to the outer circle at M. Let l intersect $\overrightarrow{AB}$ at P and $\overrightarrow{AC}$ at Q. Prove that $AP + PM = AQ + QM$.

Part v: Describe how to construct the line l of part iv. Using the previous parts perform the following on a large sheet of blank paper: construct a segment $\overline{PQ}$ with P on $\overrightarrow{AB}$ and Q on $\overrightarrow{AC}$ which passes through M such that $AP + PM = AQ + QM$. Use a ruler to check how well the construction works in practice.

April 1994

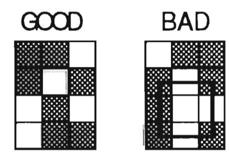
The Mandelbrot Competition

Division A Round Five Team Test

Facts: One is commonly interested in choosing two objects from among a total of n different objects; the number of ways in which this can be done is denoted $\binom{n}{2}$ and is given by the handy formula $\binom{n}{2} = \frac{n(n-1)}{2}$. For example, given the four letters A, B, C, and D, there are exactly six ways to choose two of them, which is what the formula predicts.

We now state the *Pigeonhole Principle.* Simply put, it says that if one has $k+1$ pigeons stuffed in any manner into k pigeon holes then there has to be a pigeon hole with at least two pigeons in it. This is basically common sense.

Setup: This team test will investigate the following problem. In an $m \times n$ grid what is the largest number of 1×1 squares that can be colored so that no four of the colored squares form the corners of a rectangle with vertical and horizontal sides? The grids to the right give examples of two 4×3 grids. The first one does not contain any rectangles formed by the shaded squares, while the second one does.



Problems:

Part i: Experiment to determine the maximum number of shaded squares possible in the 3×3, 4×4, and 5×5 cases. Include grids with your solutions.

Part ii: Suppose that one is given an $m \times n$ grid with some of the squares already shaded. Let a_k denote the number of squares in the k^{th} row that are shaded. Prove that if

$$\binom{a_1}{2} + \binom{a_2}{2} + \cdots + \binom{a_m}{2} > \binom{n}{2}$$

then there must be a rectangle created by four of the colored squares.

We now wish to shade a total of T squares (which means $T = a_1 + a_2 + \cdots + a_m$) in our $m \times n$ grid without creating any rectangles. Since a rectangle is necessarily formed if $\binom{a_1}{2} + \binom{a_2}{2} + \cdots + \binom{a_m}{2} > \binom{n}{2}$, we will want to minimize the left hand side.

Part iii: Show that if we have $a_i \geq a_j + 2$ for some i and j then replacing a_i by $a_i - 1$ and a_j by $a_j + 1$ won't affect T but will reduce $\binom{a_1}{2} + \binom{a_2}{2} + \cdots + \binom{a_m}{2}$.

Part iv: Use the idea of part iii to argue that the left hand side is minimized when $|a_i - a_j| = 0$ or 1; in other words when the numbers $a_1, \ldots, a_m$ are as close together as possible. Compute this minimum for $m = n = 6$, $T = 17$ and for $m = n = 7$, $T = 22$.

Part v: Prove that the maximum number of squares that can be colored in a 6×6 grid without forming rectangles is 16, and the maximum for the 7×7 case is 21. Include a diagram showing how to achieve the claimed maximum number of squares for both cases.

April 1994

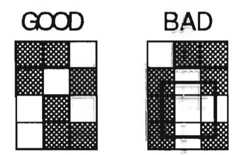
The Mandelbrot Competition

Division B Round Five Team Test

Facts: One is commonly interested in choosing two objects from among a total of n different objects; the number of ways in which this can be done is denoted $\binom{n}{2}$ and is given by the handy formula $\binom{n}{2} = \frac{n(n-1)}{2}$. For example, given the four letters A, B, C, and D, there are exactly six ways to choose two of them, which is what the formula predicts.

We now state the *Pigeonhole Principle.* Simply put, it says that if one has $k+1$ pigeons stuffed in any manner into k pigeon holes then there has to be a pigeon hole with at least two pigeons in it. This is basically common sense.

Setup: This team test will investigate the following problem. In an $m \times n$ grid what is the largest number of 1×1 squares that can be colored so that no four of the colored squares form the corners of a rectangle with vertical and horizontal sides? The grids to the right give examples of two 4×3 grids. The first one does not contain any rectangles formed by the shaded squares, while the second one does.



Problems:

Part i: Experiment to determine the maximum number of shaded squares possible in the 3×3, 4×4, and 5×5 cases. No proofs necessary, just include grids with your solutions.

Part ii: Suppose we are given a 7×7 grid with some of the squares shaded. Let a_k denote the number of colored squares in the k^{th} row for $k = 1, 2, \ldots, 7$. Show that there are $\binom{a_1}{2}$ *pairs* of columns in which both columns contain a shaded block from the first row.

Part iii: Continue this reasoning to show that there are $\binom{a_1}{2} + \binom{a_2}{2} + \cdots + \binom{a_7}{2}$ instances altogether where a pair of columns intersects some row at shaded blocks. However there are 7 columns, and hence exactly $\binom{7}{2}$ pairs of columns. Use the Pigeonhole Principle to prove that if $\binom{a_1}{2} + \binom{a_2}{2} + \cdots + \binom{a_7}{2} > \binom{7}{2}$, then there must exist four shaded squares that form a rectangle.

We now wish to shade a total of T squares (which means $T = a_1 + a_2 + \cdots + a_7$) in our 7×7 grid without creating any rectangles. Since a rectangle is necessarily formed if $\binom{a_1}{2} + \binom{a_2}{2} + \cdots + \binom{a_7}{2} > \binom{7}{2}$, we will want to minimize the left hand side.

Part iv: It turns out that the left hand side is minimized when $|a_i - a_j| = 0$ or 1 for all $1 \leq i < j \leq 7$. Demonstrate that you understand what this condition means by minimizing $\binom{a_1}{2} + \binom{a_2}{2} + \cdots + \binom{a_7}{2}$ if the total number of shaded squares is 22.

Part v: We are ready to solve the problem for the 7×7 grid. Combine parts iii and iv to prove that there is no way to color 22 of the squares without forming a rectangle. Now find a way to successfully color 21 squares. And You're Done!

Divisions A and B Round One Team Test November 1993

A–Parts i,ii B–Parts i,ii,iii: Going on the principle that doing the most elaborate case means that we can probably do all the cases, we will compute C_3 first. Labeling our cars 10, 20, and 30 from slowest to fastest we make a table of the six possible arrangements with the clumps that form and the number of clumps. The cars are moving to the right.

$\longrightarrow$

Orders:	10|20|30	10|30 20	20 10|30	20 30 10	30 10|20	30 20 10
# Clumps:	3	2	2	1	2	1

Therefore the total number of clumps is $C_3 = 3+2+2+1+2+1 = 11$ for the six different orderings. To determine the expected number of clumps we divide the total number of clumps in all the orderings by the number of orderings to find that $E_3 = \frac{C_3}{3!} = \frac{11}{6}$. In fact $1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$ so our findings agree with the formula given in the setup section. The cases $n = 1$ and 2 are even simpler. Constructing a table similar to the one above we find that $C_1 = 1$ and $C_2 = 3$, hence the expected number of clumps are $E_1 = 1$ and $E_2 = \frac{3}{2}$ respectively, as predicted by the formula.

Suppose that we have n cars in some given arrangement which form k clumps. We claim that inserting another ($10(n+1)$ mph) car faster than any of the original n cars results in either k or $k+1$ clumps depending on whether or not the new car is added at the front of the line. Suppose that we insert the $10(n+1)$ mph car behind some other car in the line. The fast car will be forced to join the clump containing the car in front of it. All the other clumps remain unchanged, so there are still k clumps in our new arrangement. However, adjoining the fast car to the head of the line creates a one car clump since the new car is faster than any car behind it. The other clumps remain unchanged, so we have $k+1$ clumps.

Now consider all $(n+1)!$ arrangements of $n+1$ cars. By definition there are a total of C_{n+1} clumps. On the other hand these $(n+1)!$ arrangements can be naturally divided up into $n+1$ groups according to where the fastest ($10(n+1)$ mph) car is located. Thus there are $n!$ orderings in which the fastest car is last; these arrangements are all possible orderings of the first n cars with the fastest car inserted at the end. By definition C_n clumps are formed by all $n!$ arrangements of the first n cars, and by the above observation, adding the fastest car at the end doesn't change the number of clumps in any of these orderings, so there are a total of C_n clumps in this first group. The same is true of any one of the n groups of arrangements in which the fastest car is not in front. A total of C_n clumps will be formed in each of these groups of arrangements. So far we have counted nC_n clumps. Now consider the final $n!$ arrangements in which the fastest car is in front. By the above observation there will be a total of $C_n + n!$ clumps in this group; C_n formed by the first n cars plus, for each of the $n!$ orderings, one extra clump created by the fastest car. In total we have

$$nC_n + (C_n + n!) = (n+1)C_n + n!$$

clumps, therefore $C_{n+1} = (n+1)C_n + n!$ as desired.

A–Parts iii,iv B–Parts iv,v: There was some confusion as to what exactly had to be shown in order to prove the statement. Here is the strategy. A sequence of numbers $\{A_1, A_2, A_3, \dots\}$ has been defined according to the formula given in the problem. Our task is to show that this sequence satisfies the recursion $A_{n+1} = (n+1)A_n + n!$ for $n \geq 1$. In other words, we have to check that substituting the values for A_n and A_{n+1} into the previous formula yields a true statement for every $n \geq 1$. There is no induction involved!

So here goes. Using the definition $A_n = n!(1 + \frac{1}{2} + \cdots + \frac{1}{n})$ we find that

$$\begin{aligned}
(n+1)A_n + n! &= (n+1)(n!(1 + \frac{1}{2} + \cdots + \frac{1}{n})) + n! \\
&= (n+1)!(1 + \frac{1}{2} + \cdots + \frac{1}{n}) + \frac{(n+1)!}{n+1} \\
&= (n+1)!(1 + \frac{1}{2} + \cdots + \frac{1}{n} + \frac{1}{n+1}) \\
&= A_{n+1}.
\end{aligned}$$

This proves that the sequence $\{A_i\}$ satisfies the recursion.

Comparing our results of the previous two problems we find that we have two sequences of numbers $\{A_n\}$ and $\{C_n\}$ which satisfy the same recursion. Moreover the first terms of each sequence are the same; in part i we found that $C_1 = 1$, and $A_1 = 1!(1) = 1$ by definition. Therefore by common sense both sequences must be the same. Technically we could employ an induction argument. We know that $A_1 = C_1$, establishing the base case. Suppose that $A_n = C_n$ for some n, then $A_{n+1} = (n+1)A_n + n! = (n+1)C_n + n! = C_{n+1}$. Therefore $C_n = A_n$ for all $n \geq 1$, giving us the formula $C_n = n!(1 + \frac{1}{2} + \cdots + \frac{1}{n})$. We can now use this expression to calculate the expected number of clumps for n cars. We divide the total number of clumps in all of the orderings, which is C_n, by the number of orderings, which is $n!$. Using our formula $C_n = n!(1 + \frac{1}{2} + \cdots + \frac{1}{n})$ this number is just $1 + \frac{1}{2} + \cdots + \frac{1}{n}$, W^5.

The whole point of this exercise was to solve the recursion that we found for C_n. This is often how one solves a recursion — figure out the first couple values, try to find a pattern, guess a general formula, and then show that this general formula satisfies your recursion and has the appropriate initial values.

A–Part v: According to our prescription for finding expected value we need to add up the sizes of all the clumps found in all $n!$ orderings of n cars and then divide this sum by the number of clumps. But the sum of all the clump sizes is just the total number of cars in all $n!$ orderings since each car is in exactly one clump, so this number is $n(n!)$. We just found the total number of clumps; it is $C_n = n!(1 + \frac{1}{2} + \cdots + \frac{1}{n})$. Therefore the expected size of a randomly selected clump is

$$\frac{n(n!)}{C_n} = \frac{n(n!)}{n!(1 + \frac{1}{2} + \cdots + \frac{1}{n}} = \frac{n}{1 + \frac{1}{2} + \cdots + \frac{1}{n}}.$$

Several schools pointed out that this is the harmonic mean of the first n natural numbers.

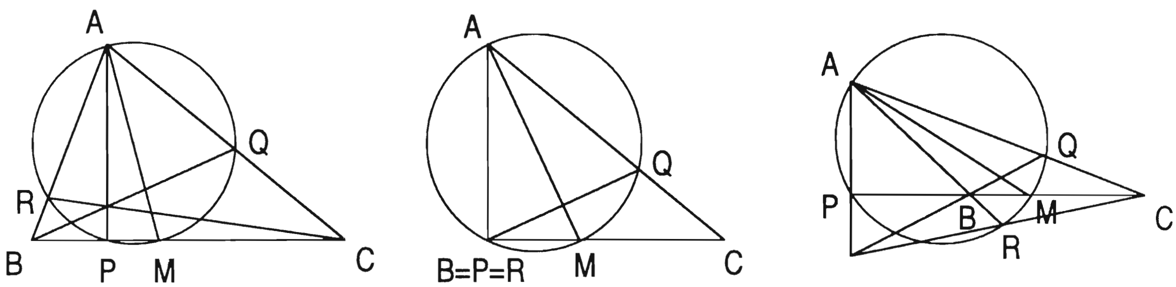
Divisions A and B — Round Two Team Test — December 1993

Before beginning I would like to address a common error that appeared in the proofs submitted by a large number of schools. It went something like this, "We need to prove an equation is true, so we will write it down here at the top, and then throw in a few facts we've figured out, and voila, we arrive at a true equation, so we're done." This is known as working backwards. *It is an excellent way to approach a problem, but it is a logically faulty proof.* Once you have successfully worked backwards on your scratch paper, start your proof with those facts and the true equation and *deduce* the equation to be proved from them.

Unlike most of the other team tests that you have taken, the five parts on this test comprise one involved geometry problem. Therefore it seems proper to write up one full proof instead of several smaller solutions. You will find that the answers to all the parts appear at some point in the proof below.

THEOREM: Let M be the foot of the angle bisector from A to side $\overline{BC}$ in $\triangle ABC$. Construct the circle with $\overline{AM}$ as diameter, and let P, Q, and R be the points of intersection of this circle with $\overleftrightarrow{BC}$, $\overleftrightarrow{AC}$, and $\overleftrightarrow{AB}$ respectively other than points A or M. Then lines $\overleftrightarrow{AP}$, $\overleftrightarrow{BQ}$, and $\overleftrightarrow{CR}$ are concurrent.

PROOF: We shall consider three cases. Either both $\angle B$ and $\angle C$ are acute, or one of them is a right angle, or one of them is an obtuse angle. These three cases are pictured below.



In all cases we will use the fact that $\angle ARM$, $\angle APM$, and $\angle AQM$ are right angles since they subtend chord $\overline{AM}$ which is a diameter. We now prove a short lemma.

LEMMA: If $\triangle ABC$ is an acute triangle then the foot of the altitude from A to $\overleftrightarrow{BC}$ lies between B and C.
PROOF: Let H be the foot of the altitude from A to $\overleftrightarrow{BC}$. Clearly neither B nor C can be this point because then $\triangle ABC$ would be a right triangle, not an acute triangle. Suppose points B, C, and H occurred along the line in that order. Then $\angle ACH$ would be obtuse since $\angle ACB$ is acute, so $\triangle ACH$ would include a right angle and an obtuse angle, which is impossible. The same problem arises if H is beyond point B to the other side, so H must lie on $\overline{BC}$. Note that the same argument demonstrates that if $\angle ABC$ is obtuse then points H, B, and C occur along line BC in that order.

Now consider $\triangle BMA$ in the first diagram. Since $\angle BAM$ is half $\angle BAC$ which is an angle in $\triangle ABC$, $\angle BAM$ must be acute. We are also given that $\angle MBA$ is acute. Since R is the foot of the altitude from M to $\overline{AB}$ and the base angles are acute as shown above, R must lie on segment $\overline{AB}$. The same logic shows that Q lies on $\overline{AC}$ and that P lies on $\overline{BC}$.

In the second case we assume without loss of generality that $\angle B$ is the right angle. Hence $m\angle ABM = 90°$ which means that B is on the circle with $\overline{AM}$ as diameter. Now by definition R is the point other than A on $\overleftrightarrow{AB}$ which intersects the circle, therefore R and B are the same point. Similarly P and B are the same point. It is now clear that in this case $\overleftrightarrow{AP}$, $\overleftrightarrow{BQ}$, and $\overleftrightarrow{CR}$ all pass through point B and thus are concurrent, so we are done in the second case.

The last case is similar to the first; we know that $\angle BAM$ is acute, $\angle MBA$ is obtuse, and R is the foot of the altitude from M to $\overleftrightarrow{AB}$. It follows that R lies on the extension of $\overline{AB}$, and in the same manner P lies on the extension of $\overline{BC}$. Therefore the diagrams above are indeed drawn correctly. (Golden rule of diagrams: never assume anything.) Some schools showed that B must be inside, on, and outside the circle in the first, second, and third diagrams respectively, which is another good way to figure out where P, Q, and R are positioned relative to A, B, and C.

Now to apply some theorems. First we argue that in all cases $AR = AQ$. Note that $\triangle ARM$ and $\triangle AQM$ are both right triangles, and furthermore $\angle RAM \cong \angle QAM$ since $\overline{AM}$ is an angle bisector. They also share side $\overline{AM}$, so $\triangle RAM \cong \triangle QAM$ by SAS, hence $AR = AQ$.

By the Power of a Point Theorem applied to B we can conclude that $(BR)(BA) = (BP)(BM)$, which can also be written $\frac{BA}{BM} = \frac{BP}{BR}$. Applying the theorem to point C we arrive at the corresponding equation $\frac{CM}{CA} = \frac{CQ}{CP}$. Multiplying these equations together yields

$$\frac{(BA)(CM)}{(BM)(CA)} = \frac{(BP)(CQ)}{(BR)(CP)}.$$

Notice that these equations hold in both the first and third cases, since the Power of a Point Theorem applies whether points B and C are inside or outside the circle.

Another theorem which holds in both cases is the Angle Bisector Theorem. This theorem tells us that if $\overline{AM}$ is an angle bisector of $\triangle ABC$ with M on $\overline{BC}$ then $\frac{AB}{BM} = \frac{AC}{CM}$, which can be rewritten as

$$\frac{(BA)(CM)}{(BM)(CA)} = 1.$$

Combining this result with the previous equation we conclude that

$$\frac{(BP)(CQ)}{(BR)(CP)} = 1.$$

Finally, we incorporate the fact that $AR = AQ$ by stating this as $\frac{AR}{AQ} = 1$ and multiplying the above equation by this one to obtain

$$\frac{(AR)(BP)(CQ)}{(AQ)(BR)(CP)} = 1.$$

We can now immediately conclude in the first case that $\overleftrightarrow{AP}$, $\overleftrightarrow{BQ}$, and $\overleftrightarrow{CR}$ are concurrent by applying Ceva's theorem as stated in the facts section of the team test. Schools that investigated Ceva's theorem hopefully also discovered that Ceva's Theorem applies even when P, Q, and R are not on the sides of $\triangle ABC$ but on their extensions! This result is sometimes called Extended Ceva, but it is really just another aspect of the same general theorem. In other words, the final equation allows us to conclude that the desired lines are concurrent in the third case as well, which completes the proof.

Divisions A and B Round Three Team Test January 1994

A–Part i B–Part i: As we shall see, the question of whether or not $p(x,y)$ is symmetric, given that $[p(x,y)]^n$ is symmetric, reduces to the question of whether or not real numbers have unique n^{th} roots. For example, suppose that $[p(x,y)]^2$ is symmetric, so that $[p(x,y)]^2 = [p(y,x)]^2$ for all pairs (x,y). Can we conclude that $p(x,y) = p(y,x)$ by taking square roots? In other words, if the squares of two real numbers are equal do the numbers themselves have to be equal? The answer is clearly no; one number could be the negative of the other. Using this idea we choose a function so that $p(x,y) = -p(y,x)$. The function $p(x,y) = x - y$ is a good counterexample, as is the function $p(x,y) = \sin(x-y)$ given in the definitions section.

On the other hand, suppose that $[p(x,y)]^3$ is a symmetric function. By definition this means that $[p(x,y)]^3 = [p(y,x)]^3$ for every pair of real numbers (x,y). This time we can conclude that $p(x,y) = p(y,x)$ by taking cube roots, since every real number has a unique cube root. (One way to convince yourself is to consider the graph of $y = x^3$.) Therefore $p(x,y)$ is symmetric.

In contrast, let $p(x,y)$ be a function of two complex variables whose output is a complex number. Then we can no longer take cube roots on both sides of the equation $[p(x,y)]^3 = [p(y,x)]^3$ because a complex number has three complex cube roots. For example, the complex number $\omega = \frac{1}{2}(-1+i\sqrt{3})$ is a cube root of one (try calculating ω^3 by hand), as is ω^2 and of course 1 itself. We can exploit the fact that there are distinct complex numbers whose cube is 1 as follows: define $p(x,y) = 1$ for all complex pairs (x,y) with $\Re(x) > 0$ and $p(x,y) = \omega$ for all other pairs (x,y) where $\Re(x) \leq 0$. Then $[p(x,y)]^3 = 1$ for all (x,y) so $[p(x,y)]^3$ is symmetric. But there are many instances where $p(x,y) \neq p(y,x)$ such as $(x,y) = (1, -1+i)$.

A–Part ii: The stipulation that $q(x,y,z)$ is symmetric "whenever it is defined" is necessary because there are bothersome examples such as $x = 0$, $y = z = \frac{\pi}{4}$ for which $q(x,y,z)$ is defined but $q(x,z,y)$ is not. Given that no zeros appear in any denominators we can calculate

$$q(x,y,z) = \frac{\sin x \cos x}{\frac{\cos y}{\sin y} + \frac{\cos z}{\sin z}} = \frac{\sin x \cos x}{\frac{\cos y \sin z + \cos z \sin y}{\sin y \sin z}} = \frac{\sin x \sin y \sin z \cos x}{\cos y \sin z + \cos z \sin y}.$$

Now note that $\cos y \sin z + \cos z \sin y = \sin(y+z)$, and using the equation $x + y + z = \frac{\pi}{2}$ we have $\sin(y+z) = \sin(\frac{\pi}{2} - x) = \cos x$, so we can simplify the above expression to arrive at

$$q(x,y,z) = \frac{\sin x \sin y \sin z \cos x}{\cos x} = \sin x \sin y \sin z.$$

This expression is clearly symmetric in x, y, and z. Since we can always write $q(x, y, z)$ in this form if it is defined we have shown that $q(x, y, z)$ is symmetric.

A–Part iii, B–Part ii: The following table lists the polynomials obtained by using the recursions defined in the set-up section.

	$f_n(x,y)$	$g_n(x,y)$	$h_n(x,y)$
$n=1$	x	y	$x+y$
$n=2$	x^2+xy	y^2	x^2+xy+y^2
$n=3$	$x^4+2x^3y+2x^2y^2+xy^3$	xy^3+y^4	$x^4+2x^3y+2x^2y^2+2xy^3+y^4$

In computing $g_3(x, y)$ we used the recursive definition $g_3(x, y) = g_2(x, y)f_2(y, x)$ and noted that $f_2(y, x) = y^2+yx$ to find $g_3(x, y) = (y^2)(y^2+yx) = xy^3+y^4$ as shown in the table. Note that none of the functions $f_n(x, y)$ or $g_n(x, y)$ is symmetric, but their sum $h_n(x, y)$ always is, at least for $n = 1$, 2, and 3.

A–Parts iv,v B–Parts iii,iv,v: Perhaps the first problem was a little easy to merit a whole question to itself. I'm sure nobody minded too much. One checks that

$$\begin{aligned} & f_n(x,y) - f_n(y,x) &=& \; g_n(y,x) - g_n(x,y) \qquad (*) \\ \iff & f_n(x,y) + g_n(x,y) &=& \; f_n(y,x) + g_n(y,x) \\ \iff & h_n(x,y) &=& \; h_n(y,x), \end{aligned}$$

using the definition $h_n(x, y) = f_n(x, y) + g_n(x, y)$. The last equation shows that $h_n(x, y)$ is symmetric.

In order to make the proof more readable we will introduce some abbreviations. Let us agree to write f_n when we mean $f_n(x, y)$ and let us write $\tilde{f}_n$ to mean $f_n(y, x)$ with the variables reversed. We will employ similar abbreviations for $g_n(x, y)$ and $h_n(x, y)$. For instance, with this notation the recursive formula for $g_{n+1}(x, y)$ can be written $g_{n+1} = g_n\tilde{f}_n$, which is much easier to deal with.

We now prove $(*)$ by induction. The base case says $f_1 - \tilde{f}_1 = \tilde{g}_1 - g_1$ which is true since both sides equal $x - y$, using the table above. Now suppose that $f_n - \tilde{f}_n = \tilde{g}_n - g_n$. By a previous argument this means that $h_n(x, y)$ is symmetric, in other words $h_n = \tilde{h}_n$. We now find that

$$\begin{aligned} f_{n+1} - \tilde{f}_{n+1} &= f_n h_n - \tilde{f}_n \tilde{h}_n && \text{(by definition)} \\ &= f_n \tilde{h}_n - \tilde{f}_n h_n && \text{(by symmetry)} \\ &= f_n(\tilde{f}_n + \tilde{g}_n) - \tilde{f}_n(f_n + g_n) && \text{(by definition)} \\ &= \tilde{g}_n f_n - g_n \tilde{f}_n && \text{(cancelling the common term)} \\ &= \tilde{g}_{n+1} - g_{n+1}. && \text{(again by definition)} \end{aligned}$$

This proves the induction step. Therefore $f_n - \tilde{f}_n = \tilde{g}_n - g_n$ for all n, and hence $h_n(x, y)$ is symmetric for all n.

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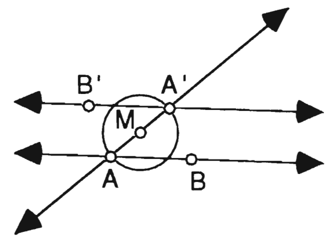
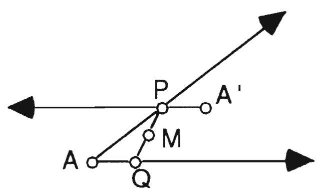
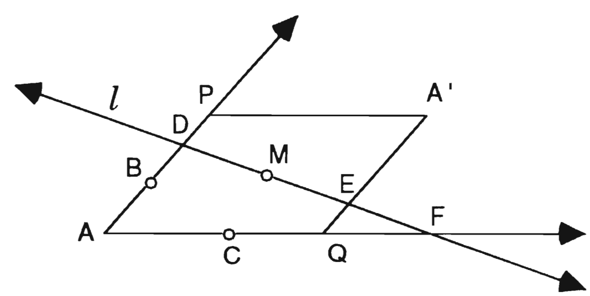
A–Part i B–Parts i,ii: We begin by describing how to construct the image of a given line under a half-turn about a given point. Label the point M and the line $\overleftrightarrow{AB}$. Since this half-turn takes $\overleftrightarrow{AB}$ to some other line we need only ascertain the images of two points on $\overleftrightarrow{AB}$ and then draw the line through them. Call the image of A the point A'. By the facts section A' is the unique point such that M is the midpoint of AA'. This suggests that we draw line AM, then construct the circle with center M and radius AM. This circle intersects $\overleftrightarrow{AB}$ in A and a second point which we define to be A'. By construction M is the midpoint of AA', so A' is the image of A. Performing the same process on B yields B', and finally drawing the line through A' and B' yields the line which is the image of $\overleftrightarrow{AB}$.





There were two popular approaches employed by students for constructing $\overline{PQ}$, each of which is instructive, so we will discuss both. In the first we construct ray $\overrightarrow{A'C'}$ which is the image of $\overrightarrow{AC}$ under a half turn about point M using the technique above. It is clear that $\overrightarrow{A'C'}$ intersects $\overrightarrow{AB}$ in a unique point which we label P. Let Q on $\overrightarrow{AC}$ be the preimage of P (the point P "came from"). A simple way to construct Q is to draw line PM and note where it intersects ray $\overrightarrow{AC}$. By the definition of half-turns M is the midpoint of $\overline{PQ}$.

The second proof proceeds as follows. Construct A' so that M is the midpoint of AA'. Next construct lines through A' that are parallel to lines AC and AB; let these intersect AB and AC in points P and Q respectively. By construction quadrilateral $APA'Q$ is a parallelogram, and we recall that the diagonals of a parallelogram bisect one another. Therefore M is the midpoint of PQ since it is the midpoint of AA'.



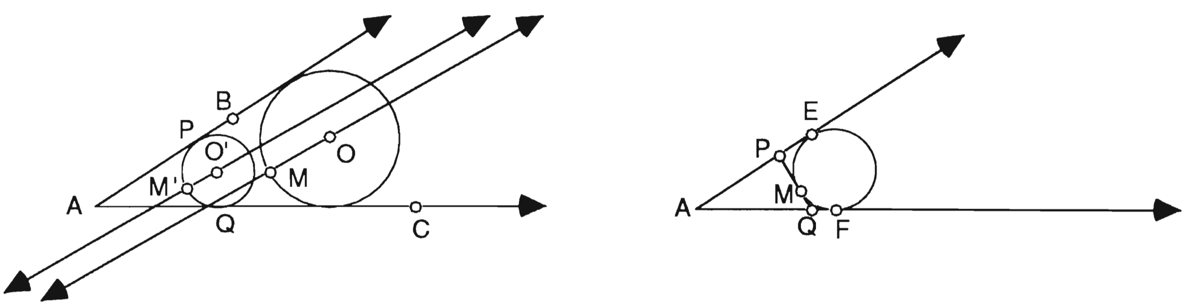
A–Part ii: We will continue the second proof just presented. Let l be any line through M which forms a triangle with $\angle BAC$, as in the diagram. Suppose l intersects $\overrightarrow{AB}$, $\overline{AQ}$, and $\overrightarrow{AC}$ in points D, E, and F respectively. Notice that l divides the area of $APA'Q$ in half. This is immediately apparent from the fact that a half turn about M takes each half to the other; P goes to Q, A goes to A', and the line l maps to itself. Since half-turns preserve lengths they preserve congruent triangles and therefore areas as well. We now see that

$$K(\triangle ADF) = K(ADEQ) + K(\triangle QEF) > K(ADEQ) = \frac{1}{2}K(APA'Q) = K(\triangle APQ),$$

where $K(\ )$ denotes the area. A similar argument applies if l is situated such that F is between A and Q, D is beyond P, and E is on $\overline{AP'}$.

A–Part iii B–Part iii: The strategy needed to solve this problem is described in the handout "An Introduction to Construction." We begin by constructing an arbitrary circle inscribed in $\angle BAC$ which doesn't necessarily pass through M. First construct the angle bisector of $\angle BAC$. Then choose an arbitrary point O' on the bisector, and drop perpendiculars from O' to $\overrightarrow{AB}$ and $\overrightarrow{AC}$ meeting these rays at P and Q. Then $\triangle APO' \cong \triangle AQO'$ by the hypotenuse-leg theorem for congruent right triangles, so $O'P = O'Q$. As a consequence the circle centered at O' passing through P also passes through Q and is tangent to both sides of the angle since $\overline{O'P}$ and $\overline{O'Q}$ are perpendiculars.

We now have a circle inscribed in $\angle BAC$; we would like to construct one which passes through point M. Draw line AM and let M' be the point of intersection of AM with the inscribed circle closer to A. The idea is that we now have a scaled down version of the solution, namely an outer circle for M'. Imagine blowing the picture back up so that M' matches up to M, then O' would scale up to the correct center O. In order to make this concrete it is useful to draw line $O'M'$ which would correspond to the parallel line OM in the bigger picture; this gives us a method for constructing O. Draw the line through O' and M' and construct a parallel line through M intersecting the angle bisector at point O. The circle with center O and radius OM is the desired circle.



A rigorous proof of this construction uses homothecy, the mathematically correct term for "scaling up" (or down). By construction triangles $AO'M'$ and AOM are similar, so the ratios $\frac{AM}{AM'}$, $\frac{AO}{AO'}$, and $\frac{MO}{M'O'}$ are equal. Performing a homothecy of the plane with center A and scale factor $\frac{AM}{AM'}$ (i.e. a scaling up by the amount $\frac{AM}{AM'}$ about A) takes M' to M and O' to O by the observation about equal ratios. Therefore it takes the circle with center O' to a circle with center O and radius $\frac{AM}{AM'}$ times as large as the old one, which is exactly the distance OM, again using the observation about equal ratios. The homothecy maps rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ to themselves so the larger circle is still tangent to both sides of $\angle BAC$. Therefore the circle we constructed above passing through M is precisely the image of the smaller circle, hence it is inscribed in $\angle BAC$ as just mentioned and we have proved that our construction works.

A–Part iv B–Parts iv,v: Most of the work has already been done in the previous problem. Given M inside $\angle BAC$ construct the outer circle as outlined above. Label the points of tangency on $\overrightarrow{AB}$ and $\overrightarrow{AC}$ as E and F respectively. Draw the line tangent to the outer circle at M. This line can be constructed relatively simply by drawing the line OM

and then constructing the perpendicular to $\overleftrightarrow{OM}$ at point M. Let this line intersect the sides of the angle at P and Q as shown. Since tangents from a point to a circle have the same length we know that $AE = AF$, $PE = PM$, and $QF = QM$. Therefore we can calculate

$$AP + PM = AP + PE = AE = AF = AQ + QF = AQ + QM.$$

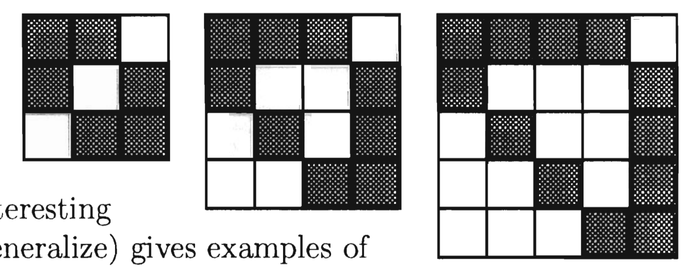
This shows that $\overline{PQ}$ is the desired segment.

A–Part v: This construction parallels the previous one very closely, so we will only outline the steps. Construct the inner circle through M, and draw the tangent to this circle at M. Let the inner circle be tangent to $\angle BAC$ at E and F, and label the points of intersection of the tangent line with rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ as P and Q. We claim that $\overline{PQ}$ is the desired segment. As before we know that $AE = AF$, $PE = PM$, and $QF = QM$. Therefore

$$AP - PM = AP - PE = AE = AF = AQ - QF = AQ - QM,$$

proving the claim.

Divisions A and B — Round Five Team Test — April 1994



A–Part i B–Part i: After trying a number of possibilities it becomes apparent that the maximum number of shaded squares that can be squeezed onto a 3×3, 4×4, and 5×5 grid are six, nine, and twelve, respectively. An interesting pattern (which unfortunately does not generalize) gives examples of grids with six, nine, and twelve shaded squares. It is not too difficult to argue that these are indeed the maximums using just the Pigeonhole Principle and some casework. However, this approach becomes tedious even for small grids. It is also possible to use the method outlined in the next four parts which is much more efficient. The reader is invited to use this method to verify the maximums stated above.

A–Part ii B–Parts ii,iii: To begin, there are $\binom{a_1}{2}$ pairs of columns which each contain a shaded block from the first row. We deduce this from the fact that there are a_1 shaded squares in the first row, and hence exactly $\binom{a_1}{2}$ ways to designate a pair of these squares. Finally, each pair of shaded squares determines a pair of columns.

We will demonstrate the 7×7 initially, the general case is no harder. Using the same reasoning as above we find that there are $\binom{a_k}{2}$ instances in which a pair of columns intersects the k^{th} row at shaded blocks. Hence there are $\binom{a_1}{2} + \binom{a_2}{2} + \cdots + \binom{a_7}{2}$ such instances altogether. Now suppose that

$$\binom{a_1}{2} + \binom{a_2}{2} + \cdots + \binom{a_7}{2} > \binom{7}{2}.$$

The left hand side counts the total number of times some pair of columns intersects some row at shaded squares. The right hand side represents the total number of pairs of columns. By the Pigeonhole Principle some pair of columns must have been counted twice in the sum on the left. In other words, some pair of columns intersects two different rows at shaded squares, which will create a rectangle.

In general we have m rows with a_k shaded squares in the k^{th} row. There are n columns and therefore $\binom{n}{2}$ pairs of columns. Suppose that

$$\binom{a_1}{2} + \binom{a_2}{2} + \cdots + \binom{a_m}{2} > \binom{n}{2}.$$

As before we conclude by the Pigeonhole Principle that some pair of columns has been counted twice in the sum on the left, which creates a rectangle.

Notice that the converse of this statement is false! We cannot conclude from

$$\binom{a_1}{2} + \binom{a_2}{2} + \cdots + \binom{a_m}{2} \leq \binom{n}{2}$$

that no rectangles are formed! For example consider $n = 5$, $m = 2$, $a_1 = 4$, and $a_2 = 3$. Then one can easily verify that these numbers satisfy the inequality above. One can also check that in a 2×5 grid with four shaded squares in the first row and three in the second that a rectangle is necessarily formed, so the converse is false.

Parts iii,iv B–Part iv: This part is a brief exercise in algebra. The following observation will simplify our calculations:

$$\binom{k}{2} - \binom{k-1}{2} = \frac{k(k-1)}{2} - \frac{(k-1)(k-2)}{2} = \frac{(k-1)[k-(k-2)]}{2} = k - 1.$$

Now suppose that $a_i \geq a_j + 2$ for some i and j. Clearly replacing a_i by $a_i - 1$ and a_j by $a_j + 1$ will not affect the sum $a_1 + \cdots + a_n$. To show that the sum $\sum \binom{a_i}{2}$ is decreased it suffices to only compare the terms involving a_i and a_j since the others are unaffected by the replacements. We find that

$$\begin{array}{rrcl} & \binom{a_i}{2} + \binom{a_j}{2} & > & \binom{a_1-1}{2} + \binom{a_j+1}{2} \\ \iff & \binom{a_i}{2} - \binom{a_1-1}{2} & > & \binom{a_j+1}{2} - \binom{a_j}{2} \\ \iff & a_i - 1 & > & a_j \\ \iff & a_i & \geq & a_j + 2, \end{array}$$

using our above calculation in the third line and the fact that all our variables are integers to obtain the fourth line. The final equation is true by hypothesis and all our steps are reversible so the initial statement is verified.

The problem dictates that the total number of shaded squares and the size of the grid remain constant. Under these constraints we will show that a necessary and sufficient condition for the quantity $\binom{a_1}{2} + \binom{a_2}{2} + \cdots + \binom{a_m}{2}$ to be minimized is that $|a_i - a_j| \leq 1$ for all $1 \leq i, j \leq m$. The previous problem shows why this condition is necessary; if $|a_i - a_j| \geq 2$

for two numbers a_i and a_j then an appropriate replacement would yield a new sequence of m integers with the same sum for which the quantity $\binom{a_1}{2} + \binom{a_2}{2} + \cdots + \binom{a_m}{2}$ is smaller, so the original sequence could not have achieved the minimum.

On the other hand there are only a finite number of possibilities for the non-negative integers $a_1, \ldots, a_m$. Therefore the minimum must be attained by one of these possibilities. There is a subtle point to realize; it is conceivable that several different sequences all satisfy the condition $|a_i - a_j| \leq 1$, but only one of them achieves the minimum. (Think about this.) We leave it as an exercise to the reader to show that there is in fact only one possible sequence (ignoring permutations) satisfying the condition whose sum is the given total T. Therefore the stated condition is also sufficient; in other words if a sequence with sum T satisfies $|a_i - a_j| \leq 1$ then it achieves the minimum.

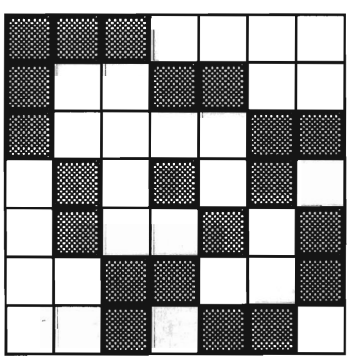
For $m = n = 6$ and $T = 17$ we find the sequence $(2,3,3,3,3,3)$ with appropriate sum satisfying our condition. Hence $\binom{2}{2} + 5\binom{3}{2} = 16$ is the corresponding minimum. Similarly when $m = n = 7$ and $T = 22$ we have the unique sequence $(3,3,3,3,3,3,4)$ of seven numbers with correct sum which satisfies $|a_i - a_j| \leq 1$. Therefore $6\binom{3}{2} + \binom{4}{2} = 24$ is the minimum.

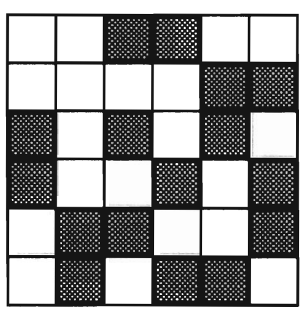
A–Part v B–Part v: We first show that a rectangle is always formed if 22 squares are shaded in a 7×7 grid. Applying the inequality just obtained we know that

$$\binom{a_1}{2} + \cdots + \binom{a_7}{2} \geq \binom{4}{2} + 6\binom{3}{2} = 24.$$

But $\binom{7}{2=21}$, which forces a rectangle to be formed. In the same manner we conclude that a rectangle will necessarily be created by shading 17 squares of a 6×6 grid, since

$$\binom{a_1}{2} + \cdots + \binom{a_6}{2} \geq 5\binom{3}{2} + \binom{2}{2} = 16 > 15 = \binom{6}{2}.$$



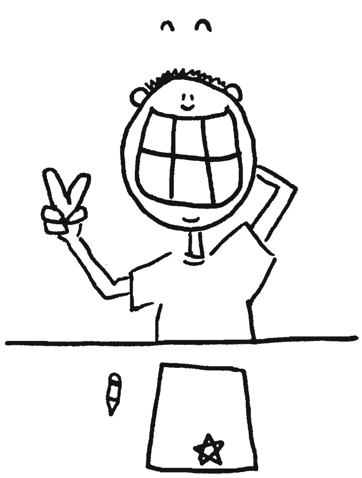


However, as pointed out the first inequality gives a sufficient but not necessary criteria for a rectangle to exist. So we must produce examples of grids with $T = 16$ and $T = 21$ to conclude that these totals can be attained. Two symmetric solutions are exhibited here; note that our 6×6 grid was created by taking the lower right block of the 7×7 grid. In general, a subgrid of a colored grid with no rectangles will also have no rectangles.

I think there are several possible paths of inquiry from here. Using Cauchy I have found a general bound of $T \leq \lfloor n(\frac{1+\sqrt{4n-3}}{2}) \rfloor$ for an $n \times n$ square. Does this expression always yield the best possible upper bound? (It does for $1 \leq n \leq 7$.) There is also the question of three dimensional grids or higher. Perhaps the reader can find other interesting results along these lines. Happy hunting.

1994-1995

The Fifth Year of the Mandelbrot Competition

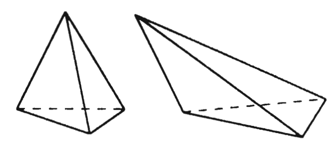


Mandelbrot Morsels

A Platonic Relationship

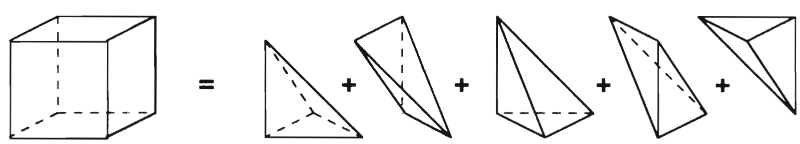
1994-95

Solid geometry is the three dimensional analogue of Euclidean geometry. Therefore the solid geometer works not only with lines, circles, and polygons but also with planes, spheres, and polyhedra. In this essay we will focus on the combinatorial aspects of solid geometry, most notably by counting vertices, edges, and faces of polyhedra. Our goal will be to enumerate all the Platonic solids using only Euler's formula, which relates the number of vertices, edges, and faces of a polyhedron. But let's start at the beginning.



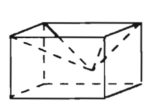
Euler's formula concerns polyhedra so we will take a moment to define them to everyone's satisfaction. The simplest polyhedron is a tetrahedron, which has four vertices which are noncoplanar (that is, not all lying in the same plane), six edges, and four triangular faces. The faces, joined along the vertices and edges, divide space into two components: the interior and exterior of the tetrahedron. Pictured above are two exemplary tetrahedra. Their interiors do not overlap while their exteriors do.

We can "glue" two tetrahedra together along congruent faces to obtain a slightly more sophisticated polyhedron. By successively gluing more tetrahedra to the growing conglomeration, always along congruent faces and in such a manner that the interiors of our tetrahedra



don't overlap, we can build up any polyhedron whatsoever. For example, the cube above can be built from five tetrahedra. We define a polyhedron to be any solid constructed in this manner as long as it is "contractible." In other words the polyhedron must loosely resemble a sphere; no polyhedra with holes (like a donut) or hidden cavities (like a soccer ball).

Before introducing Euler's formula we pause to explain some concepts involving convexity. If a polyhedron has the property that, given any two points in its interior, the segment joining them also lies in the interior, then we say the polyhedron is *convex*. A convex polyhedron doesn't have any exterior space "jutting into it" like the polyhedron to the left.



Now suppose we are given a finite set of points in space. Imagine a large balloon surrounding all the points which is allowed to deflate as much as possible. The balloon will stretch taut about exactly those points in the *convex hull*. We call the resulting polyhedron the *outer polyhedron*. As an example consider four noncoplanar points which are the vertices of a tetrahedron. Let us place a fifth point on one face of the tetrahedron and a sixth point in the interior of the tetrahedron. Then the first five points are on the convex hull, while the sixth is not. The tetrahedron is the outer polyhedron. In general given $n \geq 4$ points in space it is possible for any number between 4 and n of them to be on the convex hull.

Here is Euler's formula:

$$v + f = e + 2,$$

where v, e, and f refer to the number of vertices, edges, and faces of the polyhedron. Euler's formula applies to any polyhedron at all, not just convex polyhedra. An informal proof turns out to be relatively simple. Take a moment to check that the formula holds for a tetrahedron. Since any polyhedron can be built from tetrahedra we just have to check that the formula remains valid after we attach an additional tetrahedron. Exercising spatial prowess the reader should be able to verify that in general attaching a tetrahedron yields two new faces, three new edges, and one new vertex. The equation remains balanced so the claim is "proved" by induction on the number of tetrahedra used in the construction of our polyhedron. There are a few special cases to worry about. For example, if one of the faces of the new tetrahedron lines up perfectly with the adjoining face of the polyhedron when it is attached then these two faces merge into one face on the resulting polyhedron, so only one new face is created overall. However, the edge between these faces disappears, so only two new edges are created. As before one new vertex is created, so the contributions from the tetrahedron still balance.

A straightforward and entertaining application of Euler's formula is to determine all the regular polyhedra, also known as Platonic solids. We require that a regular polyhedron have faces which are congruent regular polygons, and also that the faces be arranged about each vertex in an identical manner. In particular each vertex must be an endpoint of the same number of edges. So let m be the number of edges meeting at each vertex. Since there are v vertices and each edge is joined to two vertices we must have $2e = mv$. Also let n be the number of edges on each face. Since each edge is part of exactly two faces we find as before that $2e = nf$. (Verify these two formulas!) Note that $m \geq 3$ and $n \geq 3$. Solving for v and f and substituting the resulting expressions into Euler's formula yields

$$e + 2 = \frac{2e}{m} + \frac{2e}{n}.$$

Clearing fractions, rearranging, and adding $4e$ to both sides we arrive at

$$\begin{aligned} emn - 2em - 2en + 4e &= 4e - 2mn. \\ \Longrightarrow e(m-2)(n-2) &= 4e - 2mn < 4e. \end{aligned}$$

Therefore $(m-2)(n-2)$ must be either 1, 2, or 3. Since m and n are positive integers greater than or equal to 3 there are only five possibilities, which are tabulated in the following chart.

Polyhedron	m	n	e	v	f
Tetrahedron	3	3	6	4	4
Cube	3	4	12	8	6
Octahedron	4	3	12	6	8
Dodecahedron	3	5	30	20	12
Icosahedron	5	3	30	12	20

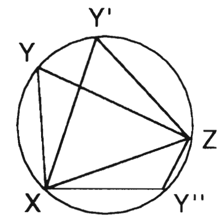
The algebra indicates that the five solutions we have found are the only possibilities. In fact there does exist one regular polyhedron for each solution, so we have found the five Platonic solids. Build colorful models of them and take them to parties.

November 1994

The Mandelbrot Competition

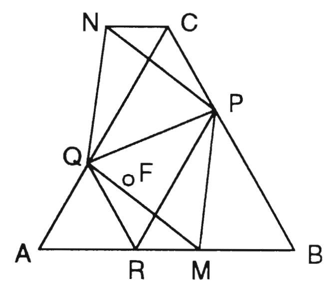
Division A Round One Team Test

Facts: We state the basic theorem on cyclic quadrilaterals. If Y and Y' are on the same side of line XZ then X, Y, Y', and Z lie on a single circle if and only if $\angle XYZ \cong \angle XY'Z$. Similarly, if Y and Y'' lie on opposite sides of line XZ then X, Y, Z, and Y'' form a cyclic quadrilateral if and only if $m\angle XYZ + m\angle XY''Z = 180°$.



Let $\triangle PQR$ be a triangle all of whose angles measure less than 120°. Then there is a point F in the interior of $\triangle PQR$ such that $m\angle PFQ = m\angle QFR = m\angle RFP = 120°$. This point is known as the Fermat point of triangle PQR.

Setup: In the diagram, triangle ABC is an equilateral triangle. Select any point R on $\overline{AB}$ and choose P and Q on $\overline{BC}$ and $\overline{AC}$ so that $\triangle ARQ$ and $\triangle BPR$ are also equilateral triangles. With base $\overline{QP}$ construct equilateral triangle QPN exterior to $\triangle QPR$ as shown. Finally, let M be the point on $\overline{AB}$ such that $AR = MB$.



Problems:

Part i: Prove that QPM is an equilateral triangle and that $MPQR$ is a cyclic quadrilateral.

Part ii: Show that $RM = NC$ and that $\overline{RM}$ is parallel to $\overline{NC}$.

Part iii: Show that rotating the plane 60° counterclockwise about point R carries Q to A and B to P. In the same manner show that $\overline{NR}$ can be obtained from $\overline{QB}$ by a 60° rotation. Use these observations to prove that $AP = BQ = NR$.

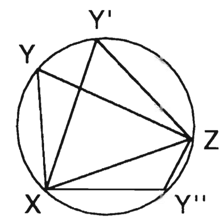
Part iv: Show that each angle of $\triangle PQR$ measures less than 120°. Prove that the Fermat point F lies on the circumcircles of triangles ARQ, BPR, and QPN.

Part v: Prove that $\overline{AP}$, $\overline{BQ}$, and $\overline{NR}$ are concurrent at F.

The Mandelbrot Competition

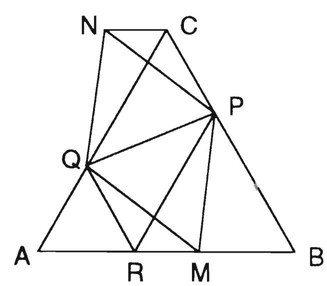
Division B Round One Team Test

Facts: We state the basic theorem on cyclic quadrilaterals. If Y and Y' are on the same side of line XZ then X, Y, Y', and Z lie on a single circle if and only if $\angle XYZ \cong \angle XY'Z$. Similarly, if Y and Y'' lie on opposite sides of line XZ then X, Y, Z, and Y'' form a cyclic quadrilateral if and only if $m\angle XYZ + m\angle XY''Z = 180°$.



A rotation of the plane through an angle α with center O is a transformation which maps each point A to a point A' such that $OA{=}OA'$ and $\angle AOA' = \alpha$. By convention positive angles denote counterclockwise rotation while negative angles indicate clockwise rotation. Rotations are isometries, which means distances are preserved; if one rotates points A and B to their images A' and B' then $AB = A'B'$. Consequently a rotation preserves lines, segments, circles, and angles. In other words, the image of a circle is a congruent circle, and similarly for the others.

Setup: In the diagram, triangle ABC is an equilateral triangle. Select any point R on $\overline{AB}$ and choose P and Q on $\overline{BC}$ and $\overline{AC}$ so that $\triangle ARQ$ and $\triangle BPR$ are also equilateral triangles. With base $\overline{QP}$ construct equilateral triangle QPN exterior to $\triangle QPR$ as shown. Finally, let M be the point on $\overline{AB}$ such that $AR = MB$.



Problems:

Part i: Show that a 60° rotation about R takes $\overline{QB}$ to $\overline{AP}$.

Part ii: Use rotations to prove that $AP = BQ = NR$.

Part iii: Prove that $PM = QM$.

Part iv: Show also that $QM = QP$, demonstrating that PQM is an equilateral triangle. Use this to prove that $PQRM$ is a cyclic quadrilateral.

Part v: Let K be the midpoint of $\overline{RC}$. By using a 180° rotation about K (or another method) prove that $NC = RM$ and that $\overline{NC}$ is parallel to $\overline{RM}$.

December 1994

The Mandelbrot Competition

Division A Round Two Team Test

Facts: The Fibonacci numbers are a remarkable sequence of integers which begin 1, 1, 2, 3, 5, 8, ...; the first two terms in the sequence are both 1 and each subsequent term is obtained by summing the two terms immediately preceding it. A more concise way to define the Fibonacci numbers is to use a *recursive definition.* Let F_n be the n^{th} Fibonacci number. Then $F_1 = 1$, $F_2 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$. There is a formula for F_n which is useful in proving identities involving Fibonacci numbers. Let $r = (1+\sqrt{5})/2$ and $s = (1-\sqrt{5})/2$ be the two roots of the polynomial $x^2 - x - 1$. Then the formula is

$$F_n = \frac{r^n - s^n}{\sqrt{5}}.$$

Note that $r + s = 1$ and $rs = -1$.

If m and n are integers expressible as a sum of two squares then their product mn can also be so written, because of the identity $(a^2+b^2)(c^2+d^2) = (ac+bd)^2 + (ad-bc)^2$.

Setup: In this team test we will consider the following question: for which positive integers a, b, and c are $ab-1$, $ac-1$, and $bc-1$ all perfect squares? In other words we are looking for solutions to the *Diophantine* equations

$$\begin{aligned} ab - 1 &= x^2 \\ ac - 1 &= y^2 \\ bc - 1 &= z^2. \end{aligned} \tag{0.1}$$

The adjective "Diophantine" means that we are only interested in integer solutions.

Problems:

Part i: Show that for $n \geq 1$ choosing $a = F_{2n-1}$, $b = F_{2n+1}$, and $c = F_{2n+3}$ produces a solution to (1).

Part ii: Recursively define two sequences of positive integers by setting $x_1 = 0$, $x_2 = 2$, $x_{n+1} = 6x_n - x_{n-1}$, and $y_1 = 1$, $y_2 = 3$, $y_{n+1} = 6y_n - y_{n-1}$. Prove that $y_n^2 = 2x_n^2 + 1$ and that $y_n y_{n-1} = 2x_n x_{n-1} + 3$.

Part iii: Using the previous result find infinitely many solutions to (1) in the special case where $a = 1$ and $b = 2$.

Part iv: Now consider the special case $a = 2$. Show that suitable b and c can be found if one can produce solutions to the Diophantine equation $(x^2+1)(y^2+1) = (2z)^2 + 4$. Use the identity in the facts section to find infinitely many solutions to this equation, and thus infinitely many solutions to (1).

Part v: Using one of the above techniques or your own find infinitely many positive integers a, b, and c such that $ab+1$, $ac+1$, and $bc+1$ are all perfect squares.

December 1994

The Mandelbrot Competition

Division B Round Two Team Test

Facts: The Fibonacci numbers are a remarkable sequence of integers which begin 1, 1, 2, 3, 5, 8, ...; the first two terms in the sequence are both 1 and each subsequent term is obtained by summing the two terms immediately preceding it. A more concise way to define the Fibonacci numbers is to use a *recursive definition*. Let F_n be the n^{th} Fibonacci number. Then $F_1 = 1$, $F_2 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$.

The Fibonacci numbers satisfy many identities, some of which you will demonstrate below. There is a formula for F_n which is useful in proving such identities. Let $r = (1+\sqrt{5})/2$ and $s = (1-\sqrt{5})/2$ be the two roots of the polynomial $x^2 - x - 1$. Then the formula can be written

$$F_n = \frac{r^n - s^n}{\sqrt{5}}.$$

Note that $r + s = 1$ and $rs = -1$.

Setup: In this team test we will consider the following question: for which positive integers a, b, and c are $ab - 1$, $ac - 1$, and $bc - 1$ all perfect squares? In other words we are looking for solutions to the *Diophantine* equations

$$\begin{aligned} ab - 1 &= x^2 \\ ac - 1 &= y^2 \\ bc - 1 &= z^2. \end{aligned} \tag{0.2}$$

The adjective "Diophantine" means that we are only interested in integer solutions.

Problems:

Part i: Verify the formula for F_n given above for $n = 1$, 2, and 3. Using this formula prove that $F_{2n-1}F_{2n+1} - 1 = F_{2n}^2$.

Part ii: In the same way show that $F_{2n-1}F_{2n+3} - 1 = F_{2n+1}^2$. Conclude that for $n \geq 1$ letting $a = F_{2n-1}$, $b = F_{2n+1}$, and $c = F_{2n+3}$ yields a solution to (1).

Part iii: Now consider the special case where $a = 2$. Show that suitable b and c can be found if one can produce solutions to the Diophantine equation $(x^2+1)(y^2+1) = (2z)^2 + 4$.

Part iv: Verify the identity $(x^2+1)(y^2+1) = (xy+1)^2 + (x-y)^2$. Use this identity to find infinitely many solutions to the equation in part three, and thus infinitely many solutions to (1).

Part v: Using one of the above techniques or your own find infinitely many positive integers a, b, and c such that $ab + 1$, $ac + 1$, and $bc + 1$ are all perfect squares.

January 1994

The Mandelbrot Competition

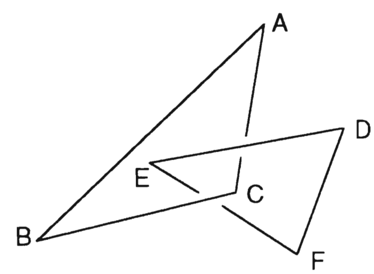
Division A Round Three Team Test

Facts: A polyhedron is the three dimensional analogue of a polygon. It is a solid object in space bounded by polygonal faces which meet one another at edges and vertices of the polyhedron. Some familiar examples of polyhedra are cubes and pyramids.

Given a finite number of points in space we can define a subset of these points called their *convex hull.* Intuitively, if one were to let a large balloon enclosing all the points slowly deflate it would eventually stretch taut about exactly those points in the convex hull. We will call the resulting solid the outer polyhedron.

Let v, e, and f denote the number of vertices, edges, and faces of an arbitrary polyhedron. Then $v + f = e + 2$; this equation is known as Euler's formula.

Setup: In the following problems we are given six points in space, no four of which lie in the same plane. Label the points A through F and consider triangles ABC and DEF in space. We say these two triangles are linked if and only if exactly one of the edges AB, AC, or BC intersects the interior of $\triangle DEF$ (or vice versa). The picture at the right illustrates linked triangles.



On this team test we will consider a surprising theorem which states that one can always divide the six given points into two groups of three points in such a way that the triangles formed by the two groups of points are linked.

Problems:
In the first four parts we will assume that the convex hull of the given points consists of all six points, so that the outer polyhedron has six vertices.
Part i: Show that all the faces of the outer polyhedron are triangles. Combine this fact with Euler's formula to show that $f = 8$ and $e = 12$.
Part ii: Let v_A denote the number of edges of the outer polyhedron having A as an endpoint, and similarly define $v_B, \ldots, v_F$. Outline an argument to show that it is impossible to have $v_A = v_B = v_C = 5$ and $v_D = v_E = v_F = 3$.
Part iii: Using the previous part conclude that some vertex lies on exactly four edges of the outer polyhedron. Let A be the point on four edges, say edges AB, AC, AD, and AE. Label the points so that the edges occur around point A in the order just listed. Show that either FB and FD or FC and FE are edges of the outer polyhedron.
Part iv: Assume that FB and FD are edges of the outer polyhedron. Argue that AF intersects the interior of $\triangle CBE$ or $\triangle CDE$ and find, with proof, a pair of linked triangles.
Part v: Using similar methods prove the theorem when five points are on the convex hull and the sixth is in the interior of the outer polyhedron.

January 1994

The Mandelbrot Competition

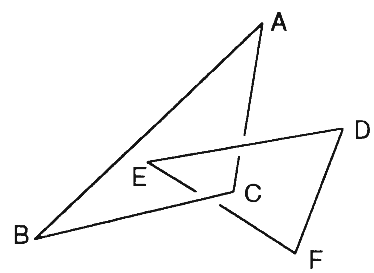
Division B Round Three Team Test

Facts: A polyhedron is the three dimensional analogue of a polygon. It is a solid object in space bounded by polygonal faces which meet one another at edges and vertices of the polyhedron. Some familiar examples of polyhedra are cubes and pyramids.

Given a finite number of points in space we can define a subset of these points called their *convex hull.* Intuitively, if one were to let a large balloon enclosing all the points slowly deflate it would eventually stretch taut about exactly those points in the convex hull. We will call the resulting solid the outer polyhedron.

Let v, e, and f denote the number of vertices, edges, and faces of an arbitrary polyhedron. Then $v + f = e + 2$; this equation is known as Euler's formula.

Setup: In the following problems we are given six points in space, no four of which lie in the same plane. Label the points A through F and consider triangles ABC and DEF in space. We say these two triangles are linked if and only if exactly one of the edges AB, AC, or BC intersects the interior of $\triangle DEF$ (or vice versa). The picture at the right illustrates linked triangles.



On this team test we will consider a surprising theorem which states that one can always divide the six given points into two groups of three points in such a way that the triangles formed by the two groups of points are linked.

Problems:

In the first four parts we will assume that five points, say A through E, are in the convex hull. Thus the outer polyhedron has five vertices with point F in its interior.

Part i: Show that all the faces of the outer polyhedron are triangles and deduce that $e = 3f/2$.

Part ii: Using Euler's formula show that $f = 6$ and $e = 9$. Argue that one of the vertices of the outer polyhedron is the endpoint of only three edges of the outer polyhedron.

Part iii: Assume without loss of generality that A is the vertex on only three edges; say AB, AC, and AD. Show that AE must lie inside the outer polyhedron but EB, EC, and ED are all edges of the outer polyhedron.

Part iv: Argue the AE must intersect the interior of one of the triangles BCF, BDF, or CDF. Assume without loss of generality that AE intersects $\triangle BDF$. Prove that triangles ACE and BDF are linked.

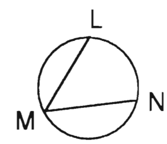
Part v: Now prove the theorem if the convex hull consists of only points A through D, so that the outer polyhedron is a tetrahedron containing E and F in its interior.

The Mandelbrot Competition

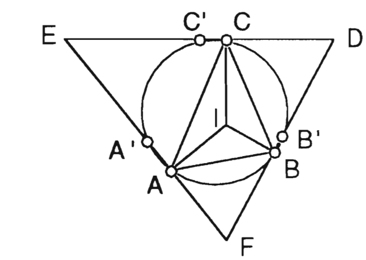
Division A Round Four Team Test

Facts: We review three of the most common points associated with a triangle ABC. The angle bisectors are concurrent at the incenter, usually denoted I. The incenter is equidistant from all three sides and hence is the center of the inscribed circle. The medians (segments joining a vertex to the midpoint of the opposite side) are concurrent at the centroid G. The triangle with the midpoints as vertices is called the medial triangle. Finally, the altitudes are concurrent at the orthocenter H. The triangle with the feet of the altitudes as vertices is known as the orthic triangle.

Recall the theorem on angles inscribed in circles. If L, M, and N are points on a circle as shown, then $\angle LMN = \frac{1}{2}\widehat{LN}$. That is, an inscribed angle equals one-half the measure of the subtended arc.



Setup: In the diagram, I is the incenter of triangle ABC. Lines through A, B, and C are constructed perpendicular to lines IA, IB, and IC respectively. These three lines form triangle DEF as labeled in the diagram. The circumcircle of $\triangle ABC$ intersects the sides of $\triangle DEF$ again in points A', B', and C' as indicated to the right. The goal of this team test will be to show that $\triangle A'B'C'$ is the medial triangle of $\triangle DEF$ and that $\triangle ABC$ is the orthic triangle of $\triangle DEF$. You may use the diagram exactly as it is pictured; in particular you can assume in your arguments that the points A, A', C', C, B', and B occur around the circle in the order pictured.



Problems:

For all the computations in the following problems find the desired angle or arc measure in terms of the measures of $\angle A$, $\angle B$, and $\angle C$. Here $\angle A$ means $\angle BAC$ and similarly.

Part i: Compute the measures of arcs $\widehat{AC'}$ and $\widehat{AC}$.

Part ii: Continue the work of the previous part by computing the measures of arcs $\widehat{A'C'}$, $\widehat{C'C}$, $\widehat{CB'}$, $\widehat{B'B}$, $\widehat{BA}$, and $\widehat{AA'}$.

Part iii: Calculate the measures of $\angle A'$, $\angle B'$, $\angle C'$, $\angle D$, $\angle E$, and $\angle F$. As before $\angle A'$ stands for $\angle B'A'C'$, $\angle D$ refers to $\angle EDF$, and similarly.

Part iv: Show that $FA'C'B'$ is a parallelogram. Using such parallelograms show that $\triangle A'B'C'$ is the medial triangle of $\triangle DEF$.

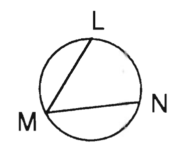
Part v: Prove that A, I, and D are collinear. By the same reasoning B, I, E and C, I, F are collinear. Show that I is the orthocenter of $\triangle DEF$ and consequently that $\triangle ABC$ is the orthic triangle of $\triangle DEF$.

The Mandelbrot Competition

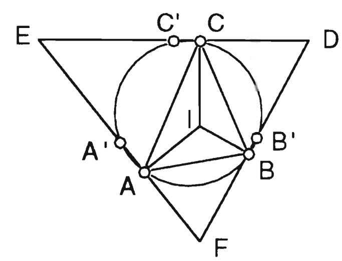
Division B Round Four Team Test

Facts: We review two of the most common points associated with a triangle ABC. The angle bisectors are concurrent at the incenter, usually denoted I. The incenter is equidistant from all three sides and hence is the center of the inscribed circle. The medians (segments joining a vertex to the midpoint of the opposite side) are concurrent at the centroid G. The triangle with the midpoints as vertices is called the medial triangle.

Recall the theorem on angles inscribed in circles. If L, M, and N are points on a circle as shown, then $\angle LMN = \frac{1}{2}\widehat{LN}$. That is, an inscribed angle equals one-half the measure of the subtended arc.



Setup: In the diagram, I is the incenter of triangle ABC. Lines through A, B, and C are constructed perpendicular to lines IA, IB, and IC respectively. These three lines form triangle DEF as labeled in the diagram. The circumcircle of $\triangle ABC$ intersects the sides of $\triangle DEF$ again in points A', B', and C' as indicated to the right. The goal of this team test will be to show that $\triangle A'B'C'$ is the medial triangle of $\triangle DEF$. You may use the diagram exactly as it is pictured; in particular you can assume in your arguments that the points A, A', C', C, B', and B occur around the circle in the order pictured.



Problems:

For all the computations in the following problems find the desired angle or arc measure in terms of the measures of $\angle A$, $\angle B$, and $\angle C$. Here $\angle A$ means $\angle BAC$ and similarly.

Part i: Show that $\widehat{AC'} = 180° - \angle C$.

Part ii: Using the previous result compute the measures of arcs $\widehat{CC'}$ and $\widehat{A'C'}$.

Part iii: In the same manner calculate the measures of arcs $\widehat{A'B'}$ and $\widehat{B'C'}$. Thus deduce the measures of angles $\angle A'$, $\angle B'$, and $\angle C'$, where $\angle A'$ refers to $\angle B'A'C'$ and similarly.

Part iv: Show that $FA'C'B'$ is a parallelogram.

Part v: Using parallelograms such as the one from part iv, show that $\triangle A'B'C'$ is the medial triangle of $\triangle DEF$.

The Mandelbrot Competition

Division A Round Five Team Test

Facts: A generating function is a way of encoding a sequence of numbers into an algebraic expression. If $\{a_0, a_1, a_2, \ldots\}$ is our given sequence of real numbers then the corresponding generating function is $f(t) = a_0 + a_1 t + a_2 t^2 + \cdots$. On this team test it is enough to understand the following simple example. We will compute the generating function for the sequence $\{1, 1, 1, 1, \ldots\}$; by definition it is $f(t) = 1 + t + t^2 + t^3 + \cdots$. This infinite sum is a convergent geometric series (when $|t| < 1$) with first term 1 and common ratio t, yielding the closed form expression $f(t) = \frac{1}{1-t}$.

Setup: An equation such as $2x + 3y = 17$ has only a finite number of solutions if we require both x and y to be nonnegative integers. In this example there are exactly three solutions. One of them is $x = 1$ and $y = 5$ which we can also state as $(x, y) = (1, 5)$. Using this notation the other solutions are $(4, 3)$ and $(7, 1)$.

Consider the following equations for some given integer $k \geq 1$.

$$x + 3y = 2k - 1, \quad 3x + 5y = 2k - 3, \quad \ldots, \quad (2k-1)x + (2k+1)y = 1.$$

Each equation has a certain number of solutions in nonnegative integers (x, y) as illustrated above. Note that we are examining each equation separately; these are *not* simultaneous equations. In this team test we will prove that the total number of solutions, obtained by adding up the number of solutions for each individual equation, is exactly k.

Problems:

Part i: Write out the seven equations for $k = 7$. Compute by hand the total number of solutions and verify the claim made in the setup section.

Part ii: Argue that the number of solutions to the equation $x + 3y = 2k - 1$ is the same as the coefficient of t^{2k-1} in the product $(1 + t + t^2 + t^3 + \cdots)(1 + t^3 + t^6 + t^9 + \cdots)$.

Part iii: Show that for a given k the total number of solutions to all the equations listed in the setup section is the same as the coefficient of t^{2k-1} in the generating function

$$\underbrace{\overbrace{(1+t+\cdots)(1+t^3+\cdots)}^{\text{first term}} + t^2(1+t^3+\cdots)(1+t^5+\cdots)}_{\text{first two terms}} + t^4(1+t^5+\cdots)(1+t^7+\cdots) + \cdots$$

Part iv: We will now find a closed form expression for the sum displayed above. Show that the sum of the first n terms is $\frac{1+t^2+\cdots+t^{2n-2}}{(1-t)(1-t^{2n+1})}$.

Part v: As n approaches infinity the partial sums found in the previous part converge to the quantity $\frac{1}{(1-t)(1-t^2)}$. Since we have merely rewritten the sum obtained in part iii the total number of solutions is still the coefficient of t^{2k-1}. By expanding $1/(1-t)$ and $1/(1-t^2)$ in geometric series show that this coefficient is equal to k, completing the proof.

April 1994

The Mandelbrot Competition

Division B Round Five Team Test

Facts: A generating function is a way of encoding a sequence of numbers into an algebraic expression. If $\{a_0, a_1, a_2, \ldots\}$ is our given sequence of real numbers then the corresponding generating function is $f(t) = a_0 + a_1 t + a_2 t^2 + \cdots$. On this team test it is enough to understand the following simple example. We will compute the generating function for the sequence $\{1, 1, 1, 1, \ldots\}$; by definition it is $f(t) = 1 + t + t^2 + t^3 + \cdots$. This infinite sum is a convergent geometric series (when $|t| < 1$) with first term 1 and common ratio t, yielding the closed form expression $f(t) = \frac{1}{1-t}$.

Setup: An equation such as $2x + 3y = 17$ has only a finite number of solutions if we require both x and y to be nonnegative integers. In this example there are exactly three solutions. One of them is $x = 1$ and $y = 5$ which we can also state as $(x, y) = (1, 5)$. Using this notation the other solutions are $(4, 3)$ and $(7, 1)$.

Consider the following equations for some given integer $k \geq 1$.

$$x + 3y = 2k - 1, \quad 3x + 5y = 2k - 3, \quad \ldots, \quad (2k-1)x + (2k+1)y = 1.$$

Each equation has a certain number of solutions in nonnegative integers (x, y) as illustrated above. Note that we are examining each equation separately; these are *not* simultaneous equations. In this team test we will prove that the total number of solutions, obtained by adding up the number of solutions for each individual equation, is exactly k.

Problems:

Part i: Write out the seven equations for $k = 7$. Compute by hand the total number of solutions and verify the claim made in the setup section.

Part ii: Argue that the number of solutions to the equation $x + 3y = 13$ is the same as the coefficient of t^{13} in the product $(1 + t + t^2 + t^3 + \cdots)(1 + t^3 + t^6 + t^9 + \cdots)$.

Part iii: Show that for $k = 7$ the total number of solutions to all the equations listed in the setup section is the same as the coefficient of t^{13} in the generating function

$$\underbrace{\overbrace{(1+t+\cdots)(1+t^3+\cdots)}^{\text{first term}}+t^2(1+t^3+\cdots)(1+t^5+\cdots)}_{\text{first two terms}}+t^4(1+t^5+\cdots)(1+t^7+\cdots)+\cdots$$

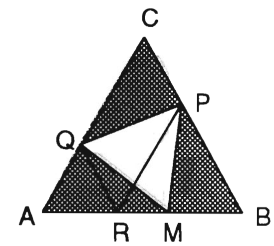
Explain why these two numbers must be equal rather than just calculating both of them.

Part iv: The sum of the first n terms of the above series is $\frac{1+t^2+\cdots+t^{2n-2}}{(1-t)(1-t^{2n+1})}$. Verify this formula for $n = 1$ and $n = 2$. You will need the formula for an infinite geometric series.

Part v: The partial sums above converge to the quantity $\frac{1}{(1-t)(1-t^2)}$. Therefore we have a closed form expression for the sum in part iii. Expand $1/(1-t)$ and $1/(1-t^2)$ in geometric series and show that the coefficient of t^{13} in the product is $k = 7$ as desired.

Divisions A and B Round One Team Test November 1994

A–Part i B–Parts iii,iv: The diagram which appeared on the team test is reproduced below in various guises for your viewing pleasure. Depending upon where R is chosen on $\overline{AB}$ point M will either lie to the right, left, or coincide with point R. In the following proofs we will assume that M is to the right of R as pictured; all the results are valid and the proofs similar in the other cases.

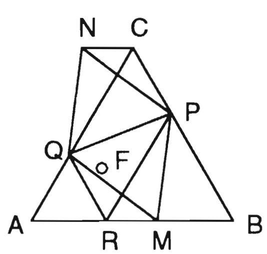


We will prove that $\triangle PQM$ is equilateral by showing that $PM = QM = PQ$. Consider triangles MAQ, PBM, and QCP. For convenience we write $AR = a$ and $RB = b$. By the given, $MB = AR = a$, thus $AM = b$ since $AB = a + b$. From the various equilateral triangles we also find that $AQ = a$ and $PB = b$. Because $AB = AC = BC = a + b$ we deduce that $QC = b$ and $CP = a$. Each of the vertex angles of $\triangle ABC$ measures 60°. In summary, each of triangles MAQ, PBM, and QCP has one side of length a, one side of length b, and an included angle of 60°. Hence all three triangles are congruent, so $PM = QM = PQ$ as desired.

Since $\triangle PQM$ is equilateral, $\angle PMQ = 60°$. But $\angle PRQ = 60°$ as well since both $\angle QRA$ and $\angle PRB$ measure 60°, and these three angles together sum to 180°. Since R and M lie on the same side of $\overline{PQ}$, $PMRQ$ is a cyclic quadrilateral by the theorem in the facts section.

A–Part ii B–Part v: Here is an elegant proof using rotations. Let O be the midpoint of $\overline{QP}$. The abundance of 60° angles shows that $CPRQ$ and $NPMQ$ are parallelograms: each of these quadrilaterals has two 60° angles and two 120° angles. Since the diagonals of a parallelogram bisect one another O must also the midpoint of $\overline{CR}$ and $\overline{NM}$. Therefore a 180° rotation about O (also known as a half turn with center O) carries R to C and M to N, and hence maps $\overline{RM}$ to $\overline{CN}$. This half turn demonstrates that $CN = RM$ since rotations preserve length, and also shows that lines CN and RM differ by a 180° angle, that is, are parallel.

One can also attack this problem directly with the basic tools of plane geometry. The overall strategy will be to prove that $\triangle PCN \cong \triangle QRM$. We first note that PQN and PQM are equilateral triangles with the common side $\overline{PQ}$. Hence all the edges of these two triangles are congruent; in particular $PN = QM$. We saw above that $CP = AQ$, and $AQ = AR$ since $\triangle AQR$ is equilateral; thus $CP = QR$. Clearly $\overline{QR} \parallel \overline{CB}$, so $\angle CPQ = \angle RQP$. Both of the angles $\angle NPQ$ and $\angle MQP$ measure 60°, so we can subtract them from our previous pair of equal angles yielding $\angle CPN = \angle RQM$. By side-angle-side we conclude that $\triangle PCN \cong \triangle QRM$. Hence $\overline{CN} \cong \overline{RM}$, finishing the first part of the problem. We then simply observe that $\angle NCP = \angle MRQ = 120°$ while $\angle PBM = 60°$ to conclude that $\overline{CN} \parallel \overline{RM}$ since the adjacent interior angles formed by the transversal $\overleftrightarrow{CB}$ sum to 180°.



A–Part iii B–Parts i,ii: Since $\triangle ARQ$ is equilateral we know that $RA = RQ$ and $\angle ARQ = 60°$. Therefore a 60° counterclockwise rotation about R carries Q to A. By the same reasoning this rotation carries B to P. Hence the rotation maps $\overline{BQ}$ to $\overline{PA}$, proving that $BQ = PA$. Using the fact that triangles PBR and PQN are both equilateral we can apply the same argument as above to show that a 60°clockwise rotation maps Q to N and B to R, thus taking $\overline{QB}$ to $\overline{NR}$. Therefore $QB = NR$, and combining this equality with the previous one we have shown that $AP = BQ = NR$.

A–Parts iv,v: We have seen above that $\angle PRQ = 60°$. Since the angles in a triangle sum to 180° we find that $\angle PQR + \angle QPR = 120°$. Since both of these angles have positive measure, each of them must be less than 120°. This means that we can apply the statement in the facts section which says that a point F exists in the interior of triangle PQR with $\angle PFQ = \angle QFR = \angle RFP = 120°$.

Since F is in the interior of $\triangle PQR$ we know that F and A lie on opposite sides of $\overleftrightarrow{QR}$. We also know that $\angle QFR = 120°$ and that $\angle QAR = 60°$, so these angles are supplementary. Applying the theorem in the facts section we conclude that Q, A, R, and F all lie on a single circle; in other words F lies on the circumcircle of $\triangle ARQ$. Precisely the same argument shows that F also lies on the circumcircles of triangles BPR and QPN.

And now for the exciting climax of the A team test: proving that the lines AP, BQ, and NR are concurrent at the Fermat point of triangle PQR. In previous years we have used Ceva's theorem as our main tool for proving concurrency of lines; on this test we will use another strategy — guessing the point at which all three lines meet and then proving that this point indeed lies on all three lines. Our guess in this case is the Fermat point of $\triangle PQR$; showing that it lies on the appropriate lines turns out to be very little work given our preparations in the first four parts.

Since $AQFR$ is a cyclic quadrilateral we deduce that $\angle AFR = \angle AQR = 60°$. We already know that $\angle PRF = 120°$, so $\angle AFP = 180°$. In other words $\angle AFP$ is a straight angle, so F lies on $\overline{AP}$ as we wanted. In the same manner F lies on $\overline{BQ}$ and $\overline{RN}$ as well, which completes the proof that $\overline{AP}$, $\overline{BQ}$, and $\overline{RN}$ are concurrent at F.

Divisions A and B — Round Two Team Test — December 1994

A–Part i, B–Parts i,ii: Verifying the formula given for F_n for small values of n is a logical way to become accustomed to working with this definition involving r and s. It is also good practice in avoiding careless mistakes! The computation for $n = 1$ is trivial. A shortcut allows us to handily polish off the case $n = 2$. Using $r+s = 1$ we find $(r^2 - s^2)/\sqrt{5} = ((r-s)/\sqrt{5})(r+s) = F_1 \cdot 1 = 1 = F_2$, as desired. Or we can use the binomial expansion to directly calculate F_3 as

$$\frac{r^3 - s^3}{\sqrt{5}} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^3 - \left(\frac{1-\sqrt{5}}{2}\right)^3\right]$$

$$
\begin{aligned}
&= \tfrac{1}{8\sqrt{5}}[(1+3\sqrt{5}+3(\sqrt{5})^2+(\sqrt{5})^3)-(1-3\sqrt{5}+3(\sqrt{5})^2-(\sqrt{5})^3)] \\
&= \tfrac{1}{8\sqrt{5}}[6\sqrt{5}+2(5\sqrt{5})] = 2 = F_3.
\end{aligned}
$$

Let us verify that Fibonacci numbers provide solutions for small values of n. When $n = 1$, we have $a = 1$, $b = 2$, and $c = 5$, in which case $ab - 1 = 1 = 1^2$, $ac - 1 = 4 = 2^2$, and $bc - 1 = 9 = 3^2$. So far so good. We next try $n = 2$, so that $a = 2$, $b = 5$, and $c = 13$. We find that $ab - 1 = 9 = 3^2$, $ac - 1 = 25 = 5^2$, and $bc - 1 = 64 = 8^2$. In every case we obtained the square of a Fibonacci number. These observations lead us to conjecture that $F_{2n-1}F_{2n+1} - 1 = F_{2n}^2$ and that $F_{2n-1}F_{2n+3} - 1 = F_{2n+1}^2$ for all $n \geq 1$. Establishing these identities would complete the problem.

The formula $F_n = (r^n - s^n)/\sqrt{5}$ given in the facts section reduces the proofs of these identities to a computation. The steps are shown below. Notice that $r^2 + s^2 = (r+s)^2 - 2(rs) = (1)^2 - 2(-1) = 3$.

$$
\begin{aligned}
F_{2n-1}F_{2n+1} - 1 &= F_{2n}^2 \\
\left(\frac{r^{2n-1} - s^{2n-1}}{\sqrt{5}}\right)\left(\frac{r^{2n+1} - s^{2n+1}}{\sqrt{5}}\right) - 1 &= \left(\frac{r^{2n} - s^{2n}}{\sqrt{5}}\right)^2 \\
\tfrac{1}{5}(r^{4n} - (rs)^{2n-1}(r^2+s^2) + s^{4n} - 5) &= \tfrac{1}{5}(r^{4n} - 2(rs)^{2n} + s^{4n}) \\
\tfrac{1}{5}(r^{4n} - (-1)(3) + s^{4n} - 5) &= \tfrac{1}{5}(r^{4n} - 2(-1)^{2n} + s^{4n}) \\
\tfrac{1}{5}(r^{4n} - 2 + s^{4n}) &= \tfrac{1}{5}(r^{4n} - 2 + s^{4n}).
\end{aligned}
$$

The five equations above are all equivalent. Since the last equation is clearly an identity, the first one is also. To prove that $F_{2n-1}F_{2n+3} - 1 = F_{2n+1}^2$ we will need the fact that $r^4 + s^4 = (r^2+s^2)^2 - 2(rs)^2 = (3)^2 - 2(-1)^2 = 7$. Otherwise the steps are practically identical to those above; the reader is invited to perform the calculation as an exercise.

A–Parts ii,iii: Since these two sequences of numbers are defined recursively it makes sense to prove the statements by induction. We quickly find that the base cases work for both claims: $1^2 = 2(0)^2 + 1$, $3^2 = 2(2)^2 + 1$, and $3 \cdot 1 = 3 = 2(2 \cdot 0) + 3$. It turns out that each of these statements depends on the other, but a neat double induction takes care of both claims at once. So we assume that $y_n^2 = 2x_n^2 + 1$ and $y_n y_{n-1} = 2x_n x_{n-1} + 3$ for all $n \leq k$ and start experimenting. We find that

$$
\begin{aligned}
y_{k+1}^2 &= (6y_k - y_{k-1})^2 \\
&= 36y_k^2 - 12y_k y_{k-1} + y_{k-1}^2 \\
&= 36(2x_k^2 + 1) - 12(2x_k x_{k-1} + 3) + 2x_{k-1}^2 + 1 \\
&= 2(36x_k^2 - 12x_k x_{k-1} + x_{k-1}^2) + 1 \\
&= 2(6x_k - x_{k-1})^2 + 1 \\
&= 2x_{k+1}^2 + 1.
\end{aligned}
$$

This takes care of the first claim. Emboldened by our initial success we now compute

$$\begin{aligned} y_{k+1}y_k &= (6y_k - y_{k-1})y_k \\ &= 6y_k^2 - y_k y_{k-1} \\ &= 6(2x_k^2+1) - (2x_k x_{k-1} + 3) \\ &= 2x_k(6x_k - x_{k-1}) + 3 \\ &= 2x_{k+1}x_k + 3. \end{aligned}$$

This completes the induction.

If $a = 1$ and $b = 2$ then our system of equations is reduced to $2 - 1 = x^2$, $c - 1 = y^2$, and $2c - 1 = z^2$. Clearly we will choose $x = 1$. Combining the last two equations by eliminating c we wind up with $z^2 = 2y^2 + 1$, which pleasantly reminds us of all the work we had to do in the previous part. Therefore we choose $y = x_n$ and $z = y_n$, which means $c = x_n^2 + 1$. In summary, we have shown that $a = 1$, $b = 2$, $c = x_n^2 + 1$, $x = 1$, $y = x_n$ and $z = y_n$ is a solution for all $n \geq 1$, so we have found another infinite family of solutions.

A–Part iv B–Parts iii,iv: The idea behind this problem is to show that in the special case $a = 2$, finding integer solutions to the system of equations $2b - 1 = x^2$, $2c - 1 = y^2$, and $bc - 1 = z^2$ is the same as finding integer solutions to the single equation $(x^2+1)(y^2+1) = (2z)^2 + 4$. Many schools demonstrated half of the link, namely, if we have a solution to the first set of equations then

$$(x^2+1)(y^2+1) = (2b)(2c) = 4(z^2+1) = (2z)^2 + 4,$$

so x, y, and z are a solution to the second equation. We are actually interested in the converse, which is almost as straightforward. So suppose that x, y, and z are integers satisfying $(x^2+1)(y^2+1) = (2z)^2 + 4$. Note that if x is even then $x^2 + 1$ is odd, while if x is odd then two divides $x^2 + 1$ but four does not. (Work this out if it is not immediately obvious to you.) Since the right hand side is a multiple of four we need x and y to both be odd, in which case $b = (x^2+1)/2$ and $c = (y^2+1)/2$ are both integers. It is then easy to check that b, c, x, y, and z are integer solutions to the first system of equations. (Do it.)

We proceed to find some solutions to the second equation. The identity $(x^2+1)(y^2+1) = (xy+1)^2 + (x-y)^2$ is a special case of the one given in the facts section, and in any case it is easy to verify. We see that

$$(xy+1)^2 + (x-y)^2 = (x^2y^2 + 2xy + 1) + (x^2 - 2xy + y^2) = x^2y^2 + x^2 + y^2 + 1 = (x^2+1)(y^2+1).$$

Therefore we want to find solutions to $(xy+1)^2 + (x-y)^2 = (2z)^2 + 4$. Both sides look suspiciously similar at this point, so we exploit this fact by choosing $x - y = 2$ and $xy + 1 = 2z$. This can be done if x and y are both odd with x two greater than y. One way to write our solution is $x = 2t + 1$, $y = 2t - 1$, and $z = 2t^2$, as you can check. This yields an infinite number of solutions, one for each value of t, and from our work above we know that each of these solutions leads to an integer solution of the original system of equations.

A–Part v B–Part v: Any of the above methods can be modified to solve the system of equations in the last problem. In this solution we will outline how to get started; you should carry out the details as practice if any of these methods is new to you.

For starters, one could prove that $a = F_{2n}$, $b = F_{2n+2}$, and $c = F_{2n+4}$ is a solution using the formula from the facts section. Alternatively, one could define sequences $y_1 = 1$, $y_2 = 3$, $y_n = 4y_{n-1} - y_{n-2}$ and $z_1 = 1$, $z_2 = 5$, $z_n = 4z_{n-1} - z_{n-2}$ and then prove that $3y_n^2 = z_n^2 + 2$ and $3y_n y_{n-1} = z_n z_{n-1} + 4$ by a double induction. Finish by finding an infinite number of solutions when $a = 1$ and $b = 3$. Another technique is to let $a = 1$ and then show that the system of equations $b + 1 = x^2$, $c + 1 = y^2$, and $bc + 1 = z^2$ has integer solutions exactly when the equation $(x^2-1)(y^2-1) = z^2-1$ does. Using the identity $(x^2-1)(y^2-1) = (xy-1)^2-(x-y)^2$ allows one to find solutions $x = t + 1$, $y = t$, and $z = t^2 + t - 1$ which correspond to $a = 1$, $b = t^2 + 2t$, and $c = t^2 - 1$. Several schools also found a set of solutions more elementary than any of those above by defining $a = t - 1$, $b = t + 1$, and $c = 4t$. Good work!

Divisions A and B — Round Three Team Test — January 1995

The type of mathematics encountered on this team test is intriguing because it combines both geometry and combinatorics. On the one hand, our arena is three dimensional Euclidean space and our polyhedra are geometric objects. However, certain aspects of these polyhedra are independent of their particular shape or size. For example, a solid polyhedron with 30 edges and 12 faces and no holes (as described in the essay accompanying this team test) must have 20 vertices. We can see this immediately from Euler's formula, which is combinatorial in nature since it only involves the *number* of vertices, edges, and faces. Imagine how difficult it would be to prove this fact by considering all the possible geometric configurations involving 30 edges and 12 faces! For this reason the proof below is as combinatorial as possible, which makes it shorter and more precise. The solutions for the A division test are contained within cases one and two, while the B division answers comprise cases two and three.

THEOREM: Given six points in space, no four of which are coplanar, it is possible to divide the points into two sets of three points each such that the triangles formed by using each set of points as vertices are linked.

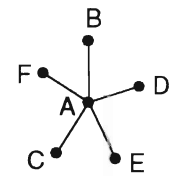
PROOF: The convex hull of the six points contains either four, five, or six points since at least four vertices are needed to create a solid. We will consider each of these cases in turn, but first note that in every case the faces of the outer polyhedron are triangles, because a face of the outer polyhedron is contained in a single plane. A face with four or more vertices would imply that these four or more points were coplanar, contradicting the hypothesis in the setup section.

Case 1: We begin by assuming that the convex hull consists of all six points, which means the outer polyhedron has six vertices. For the outer polyhedron denote the number of vertices by v, edges by e, and faces by f. If we count the number of edges around all the faces we obtain $3f$, since all faces are triangular. Each edge is included twice in this count, since each edge is part of exactly two faces, thus $e = 3f/2$. We have $v = 6$, so by Euler's formula we find

$$v + f = e + 2 \implies 6 + f = \frac{3f}{2} + 2 \implies f = 8,$$

and therefore $e = 3f/2 = 12$.

The strategy for the next segment of the proof is due to Naperville North High School. Let v_A denote the number of edges emanating from vertex A, and similarly for v_B through v_F. First note that if $v_A = 5$ then A must be connected to every other vertex of the outer polyhedron since there are only five remaining vertices. The same is true of vertices B and C. Consider two adjacent edges emanating from A, such as $\overline{AB}$ and $\overline{AD}$ in the figure to the right.



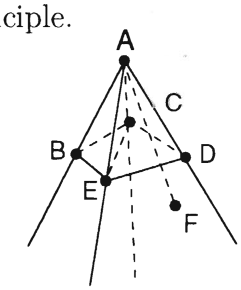
These two are edges of a face of the polyhedron, which must be a triangle by part i. Thus $\overline{BD}$ is the third edge, in particular $\overline{BD}$ is an edge of the polyhedron. Since D, E, and F are three of the five vertices joined to A two of them must be adjacent, say D and E. But then D is connected to E as well as to A, B, and C, contradicting the fact that $v_D = 3$.

Clearly $3 \leq v_A \leq 5$ since at least three edges must meet at A to form a solid angle; in addition there are at most five other vertices to which A may be connected. The same is true for the numbers v_B through v_E. We also observe that

$$v_A + v_B + v_C + v_D + v_E + v_F = 2e = 24$$

because in summing all the edges coming into each vertex we count every edge exactly twice (each edge has two endpoints) and our polyhedron contains 12 edges. If no vertex lies on 4 edges then each v_i is either 3 or 5. Consequently the only way for the above sum to equal 24 is by writing $3 + 3 + 3 + 5 + 5 + 5 = 24$, which cannot occur by the previous part. Hence some vertex lies on 4 edges, say A. Without loss of generality A is connected to B, C, D, and E. Since $v_F \geq 3$ and F is not joined to A it must be connected to at least three of B, C, D, and E; thus to either B and D or C and E by the Pigeonhole Principle.

Now we will explicitly use the convexity of the outer polyhedron. Specifically, if $\overline{AF}$ is not an edge of the polyhedron then it must lie in the interior. Therefore the part of segment AF near A lies inside the solid angle at A which is composed of the tips of the two tetrahedra $ACEB$ and $ACED$. The segment can't lie along plane ACE or else the four points A, C, E, and F would be coplanar; so it lies inside one of these two tetrahedra, say $ACED$. Finally, $\overline{AF}$ must extend beyond the base of this tetrahedron or else F would be in the interior of the polyhedron.



Therefore $\overline{AF}$ intersects the interior of $\triangle CDE$. We now claim that triangles ABF and CDE are linked. We have just shown that AF intersects $\triangle CDE$, and as AB and FB are edges of the polyhedron they cannot possibly intersect $\triangle CDE$, so by definition we have found a pair of linked triangles. The other possibility is that $\overline{AF}$ lies inside tetrahedron $ACEB$, in which case we would find analogously that triangles ADF and CBE were linked.

Case 2: In this case the convex hull consists of five points, so the outer polyhedron has five vertices with the sixth point in its interior. Denote the number of vertices by v, edges by e, and faces by f. If we add together the number of edges around each face we obtain $3f$, since each face has three sides. Every edge is included twice in this count, since each edge is part of exactly two faces. Combining these observations yields $e = 3f/2$. We are given that $v = 5$, so by Euler's formula we can deduce

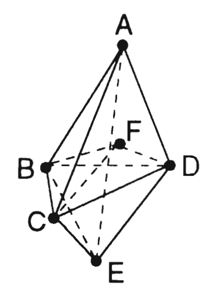
$$v + f = e + 2 \quad \Longrightarrow \quad 5 + f = \frac{3f}{2} + 2 \quad \Longrightarrow \quad f = 6.$$

It now follows that $e = 3f/2 = 9$.

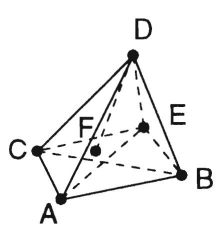
Note that every vertex must be joined to at least three other vertices by edges of the outer polyhedron in order to form a three dimensional solid angle. Of course, no vertex is the endpoint of more than four edges since there are only four remaining vertices on the outer polyhedron. If every vertex were the endpoint of four edges we would have a total of 20 endpoints, or 10 edges altogether. This contradicts the fact that $e = 9$, so at some vertex only three edges meet.

We assume that A is the vertex at which only three edges meet, say edges $\overline{AB}$, $\overline{AC}$, and $\overline{AD}$. Therefore $\overline{AE}$ is a segment between two vertices of the outer polyhedron which is not an edge. Since the outer polyhedron is convex this segment must lie in the interior. We just noted that every vertex is the endpoint of at least three edges, and since $\overline{AE}$ is not an edge E must connect to all of B, C, and D.

We can now deduce the general configuration of the five vertices. There are three edges ending at each of A and E. Since $e = 9$ there are three edges remaining, which must be $\overline{BC}$, $\overline{BD}$, and $\overline{CD}$. Clearly A and E cannot both be on the same side of the plane through $\triangle BCD$, for if they were either AE would be an edge of the outer polyhedron or one of A and E would be within the outer polyhedron, contradicting what we know. Therefore the outer polyhedron must appear as pictured below.



The part of segment AE near A lies inside the solid angle at A which is composed of the tips of the three tetrahedra $ABCF$, $ABDF$, and $ACDF$. The segment can't lie along one of the faces of these tetrahedra, such as $\triangle ACF$; or else four points would be coplanar, in this case the four points A, C, E, and F. Therefore it lies inside one of these three tetrahedra, say $ABDF$. Finally, $\overline{AE}$ must extend beyond the base of this tetrahedron or else E would be in the interior of the polyhedron. Therefore $\overline{AE}$ intersects the interior of $\triangle BDF$. We now claim that triangles ACE and BDF are linked. We have just shown that AE intersects $\triangle BDF$, and as AC and CE are edges of the outer polyhedron they cannot possibly intersect $\triangle BDF$, so by definition we have found a pair of linked triangles.



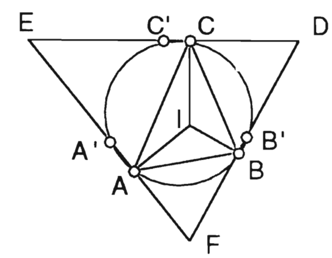
Case 3: The techniques employed in this case are reminiscent of those above, so we will streamline the argument. Since the convex hull contains only four points the outer polyhedron must be a tetrahedron. We will assume that points E and F are in the interior of the tetrahedron. Imagine drawing segments $\overline{EA}$, $\overline{EB}$, $\overline{EC}$, and $\overline{ED}$ which subdivide the outer tetrahedron into four smaller tetrahedra. As before, point F cannot lie on any of the faces of these tetrahedra since no four points are coplanar. Suppose that F lies in the interior of $ABCE$. In the same manner as the previous problem we see that $\overline{DF}$ must intersect the interiors of one of the triangles ABE, ACE, or BCE; suppose it intersects $\triangle ACE$. We claim that triangles ACE and BDF are linked. We have just seen

that $\overline{DF}$ intersects $\triangle ACE$. Segment BD cannot intersect $\triangle ACE$ since $\overline{BD}$ is an edge of the outer tetrahedron, and segment BF likewise cannot intersect $\triangle ACE$ as BF lies in the interior of $ABCE$ while $\triangle ACE$ is a face of this tetrahedron.

Notice that the arguments above are perfectly general; no matter in which tetrahedron F was located, or what face of that tetrahedron was intersected by the segment joining F to the opposite vertex, the same arguments would produce a pair of linked triangles. Only the letters would change.

Just for fun, here's another mind-bender for the stalwart problem solvers taking our competition. Clearly it is impossible to draw two nonintersecting loops on the surface of a sphere which are linked. However, it is possible to do so on the surface of a doughnut. How?

Divisions A and B — Round Four Team Test — March 1995



A–Parts i,ii,iii B–Parts i,ii,iii: The key to this type of computation is to find a useful angle which subtends the given arc. For example, $\angle ABC$ subtends $\widehat{AC}$. Therefore, by the theorem on inscribed angles, we conclude that $m\widehat{AC} = 2m\angle ABC$, which we write simply as $m\widehat{AC} = 2m\angle B$. Similarly we conclude that $m\widehat{AC'} = 2m\angle ACC'$. In order to write this in terms of $m\angle A$, $m\angle B$, and $m\angle C$ we note that

$$m\angle ACC' = 90° - m\angle ACI = 90° - \tfrac{1}{2}m\angle C,$$

so that $m\widehat{AC'} = 2(90° - \frac{1}{2}m\angle C) = 180° - m\angle C$. Since $m\angle A + m\angle B + m\angle C = 180°$ we can also write $m\widehat{AC'} = m\angle A + m\angle B$. Using exactly the same reasoning we find that $m\widehat{CA'} = m\angle B + m\angle C$ and $m\widehat{CB'} = m\angle A + m\angle C$. (Check these as practice.)

There are many ways to go about finding the remaining arc measures. We find immediately that $m\widehat{AC} = 2m\angle B$ and $m\widehat{AB} = 2m\angle C$ since angle $\angle B$ subtends arc $\widehat{AC}$ and analogously for the second equation. Hence,

$$m\widehat{CC'} = m\widehat{AC} - m\widehat{AC'} = 2m\angle B - (m\angle A + m\angle B) = m\angle B - m\angle A.$$

In the same manner $m\widehat{AA'} = m\angle B - m\angle C$ and $m\widehat{BB'} = m\angle A - m\angle C$. We can now calculate

$$\begin{aligned} m\widehat{A'C'} &= m\widehat{CA'} - m\widehat{CC'} \\ &= (m\angle B + m\angle C) - (m\angle B - m\angle A) \\ &= m\angle A + m\angle C. \end{aligned}$$

There are several equivalent answers which may include a summand of 180° or 360°. These can be simplified to the ones given above by using the fact that $m\angle A + m\angle B + m\angle C = 180°$.

We can now use the above results to find the measures of other angles in the diagram. Since $\angle B'A'C'$ intercepts arc $\widehat{B'C'}$ we discover that

$$m\angle B'A'C' = \tfrac{1}{2}m\widehat{B'C'} = \tfrac{1}{2}((m\angle B - m\angle A) + (m\angle A + m\angle C)) = \tfrac{1}{2}(m\angle B + m\angle C),$$

where we broke arc $\widehat{B'C'}$ up into the smaller arcs $\widehat{CB'}$ and $\widehat{CC'}$ which we had already measured. One finds that $m\angle B' = \frac{1}{2}(m\angle A + m\angle C)$ and $m\angle C' = \frac{1}{2}(m\angle A + m\angle B)$, a neat symmetric result. As extra practice, try to obtain these last two equations for yourself.

We can again utilize our calculations from above to find $m\angle D$ by recalling that the measure of the angle formed by two secants meeting outside a circle is the difference of the two intercepted arcs. In other words

$$\begin{aligned} m\angle D &= \tfrac{1}{2}((m\widehat{A'C'} + m\widehat{AA'} + m\widehat{AB}) - (m\widehat{CB'})) \\ &= \tfrac{1}{2}(m\angle A + m\angle C + m\angle B - m\angle C + 2m\angle C - m\angle A - m\angle C) \\ &= \tfrac{1}{2}(m\angle B + m\angle C). \end{aligned}$$

A corresponding computation yields $m\angle E = \frac{1}{2}(m\angle A + m\angle C)$ and $m\angle F = \frac{1}{2}(m\angle A + m\angle B)$. An alternative approach is to compute $\angle D$ via triangle BCD, since it is possible to calculate angles $\angle DCB$ and $\angle DBC$. Query: Do we arrive at the same expression?

A–Part iv B–Parts iv,v: Half of the work was completed in the previous section, where we found that both of $m\angle C'$ and $m\angle F$ were equal to $\frac{1}{2}(m\angle A + m\angle B)$. We need only show that the other pair of opposite angles in quadrilateral $FA'C'B'$ are equal to conclude that we have a parallelogram. Once again, those arc measures will do the job for us. Noting that $\angle FA'C' = \angle AA'C'$ we use the inscribed angle theorem to obtain

$$m\angle FA'C' = \tfrac{1}{2}(m\widehat{AB} + m\widehat{BB'} + m\widehat{CB'} + m\widehat{CC'}) = \tfrac{1}{2}(m\angle A + m\angle B + 2m\angle C).$$

In the same manner we find that $m\angle FB'C' = \frac{1}{2}(m\angle A + m\angle B + 2m\angle C)$ after a little algebra, so $FA'C'B'$ is indeed a parallelogram. It can be shown that $DB'A'C'$ and $EA'B'C'$ are also parallelograms by using an argument analogous to the one just presented.

Up to this point we have been engaged in what is informally known as "angle chasing." Having pinpointed several parallelograms through chasing angles, we will now take advantage of the fact that opposite pairs of sides of parallelograms are congruent. In particularly we find that $FB' = A'C'$ and that $DB' = A'C'$ from parallelograms $FA'C'B'$ and $DB'A'C'$ respectively. Thus $DB' = FB'$, so B' is the midpoint of segment $\overline{DF}$. (Who would have guessed that we could have established a midpoint by chasing angles?) Similarly A' and C' are the midpoints of $\overline{EF}$ and $\overline{DE}$, demonstrating that $\triangle A'B'C'$ is the medial triangle of $\triangle DEF$.

A–Part v: The solution to this problem relies on techniques developed on an earlier team test involving cyclic quadrilaterals. Our strategy for showing that A, I, and D are collinear will also be reminiscent of that test: we will prove that angle $\angle AID = 180°$.

Consider quadrilateral $DBIC$. Angles $\angle DCI$ and $\angle DBI$ are both right angles, and therefore sum to $180°$, which implies that $DBIC$ is a cyclic quadrilateral. Consequently $\angle BID = \angle BCD$, and we can compute $m\angle BCD = 90° - \frac{1}{2}m\angle C$ as before. On the other hand, $FAIB$ is a cyclic quadrilateral as well, which means that angles $\angle AIB$ and $\angle AFB$ are supplementary, so that $m\angle AIB = 180° - m\angle F = 180° - \frac{1}{2}(m\angle A + m\angle B)$. Therefore

$$\begin{aligned} m\angle AIB + m\angle BID &= 180° - \tfrac{1}{2}(m\angle A + m\angle B) + 90° - \tfrac{1}{2}m\angle C \\ &= 270° - \tfrac{1}{2}(m\angle A + m\angle B + m\angle C) \\ &= 180°, \end{aligned}$$

proving that A, I, and D are collinear.

The remainder of the problem follows quickly. By hypothesis $\overline{IC}$ is perpendicular to $\overline{ED}$, and since we have just shown that IC and FC are in fact the same line, we conclude that $\overline{FC}$ is an altitude of $\triangle DEF$. In the same manner $\overline{DA}$ and $\overline{EB}$ are also altitudes. Hence, by definition, $\triangle ABC$ is the orthic triangle of $\triangle DEF$. Finally, since I is the common point to all the altitudes, I is the orthocenter of $\triangle DEF$, polishing off the last proposition.

Divisions A and B — Round Five Team Test — April 1995

A–Part i B–Part i: Here are the seven equations, along with all solutions in nonnegative integers.

Equation	Solutions
$x + 3y = 13$	(1,4); (4,3); (7,2); (10,1); (13,0)
$3x + 5y = 11$	(2,1)
$5x + 7y = 9$	$\emptyset$
$7x + 9y = 7$	(1,0)
$9x + 11y = 5$	$\emptyset$
$11x + 13y = 3$	$\emptyset$
$13x + 15y = 1$	$\emptyset$

The symbol $\emptyset$ stands for the empty set, meaning no solutions exist. There are a total of seven solutions for $k = 7$, as predicted.

A–Parts ii,iii B–ii,iii: It is not too difficult to multiply $(1 + t + t^2 + t^3 + \cdots)$ and $(1 + t^3 + t^6 + t^9 + \cdots)$ together longhand. A pattern appears after the first couple terms which convinces us that the product will be

$$1 + t + t^2 + 2t^3 + 2t^4 + 2t^5 + 3t^6 + 3t^7 + 3t^8 + 4t^9 + \cdots.$$

For example, the coefficient of t^{13} is 5 by continuing the pattern above. Since there were 5 solutions to the equation $x + 3y = 13$, the claim in part ii is true for $k = 7$.

However, there is another way of attacking the problem which gives some insight into *why* these two numbers should be equal, and has the advantage of generalizing to complete part iii as well. After a little tinkering it becomes intuitively clear that we get a t^{2k-1} term in the product whenever a t^x and t^{3y} term are multiplied together with $x + 3y = 2k - 1$, so we're done. Now to just write this down in a coherent and rigorous manner... A useful strategy when faced with such a dilemma is to establish a one-to-one correspondence between the two sets being compared. We shall adopt this tactic in the proof as an illustration.

The coefficient of t^{2k-1} in the product $(1+t+t^2+\cdots)(1+t^3+t^6+\cdots)$ equals the number of ways to choose a term from the first sum and a term from the second sum which multiply to t^{2k-1}, since the coefficients of all these terms are 1. We claim this number is the same as the number of solutions in nonnegative integers to the equation $x+3y=2k-1$. For suppose the product of t^a and t^{3b} is t^{2k-1}. Then $t^a t^{3b} = t^{a+3b} = t^{2k-1}$ which implies $a+3b=2k-1$. Thus (a,b) is a solution to our Diophantine equation. On the other hand, let (a,b) be a solution of $x+3y=2k-1$. Then clearly the terms t^a and t^{3b} multiply to t^{2k-1}. We have succeeded in pairing each solution (a,b) with a pair of terms t^a and t^{3b} whose product is t^{2k-1}. This is the desired one-to-one correspondence which proves our claim.

Using precisely the same reasoning we find that the coefficient of t^{2k-3} in the product $(1+t^3+t^6+\cdots)(1+t^5+t^{10}+\cdots)$ equals the number of solutions in nonnegative integers to the equation $3x+5y=2k-3$. This is clearly the same as the coefficient of t^{2k-1} in the product $t^2(1+t^3+t^6+\cdots)(1+t^5+t^{10}+\cdots)$, since multiplying by t^2 just shifts all the coefficients over two places. In the same way, the number of solutions to $5x+7y=2k-5$ equals the coefficient of t^{2k-1} in the product $t^4(1+t^5+t^{10}+\cdots)(1+t^7+t^{14}+\cdots)$, and so on. The reason we shift the positions of the coefficients around with factors like t^2 and t^4 is to shift all the coefficients we are interested in onto t^{2k-1}. Therefore when we add all the terms of the generating function together, the total number of solutions will be exactly the coefficient of t^{2k-1}. Note that all the terms in the generating function beyond the k^{th} don't contribute to t^{2k-1} because the power of t in those terms is at least t^{2k}. To see these arguments work out concretely, substitute 7 for k everywhere in the last two paragraphs.

A–Part iv B–Part iv: To begin we write the first two terms in closed form by summing the convergent geometric series. We know that $1+t+t^2+t^3+\cdots = \frac{1}{1-t}$. Replacing t by t^3 or t^5 we also find that

$$1+t^3+t^6+t^9\cdots = \frac{1}{1-t^3} \quad \text{and} \quad 1+t^5+t^{10}+t^{15}+\cdots = \frac{1}{1-t^5}.$$

Therefore the first term equals $\frac{1}{(1-t)(1-t^3)}$, which is what the given formula predicts for $n=1$. Using the above closed form expressions we can add the first two terms to obtain

$$\begin{aligned}\frac{1}{(1-t)(1-t^3)} + \frac{t^2}{(1-t^3)(1-t^5)} &= \frac{1-t^5+t^2(1-t)}{(1-t)(1-t^3)(1-t^5)} \\ &= \frac{1+t^2-t^3-t^5}{(1-t)(1-t^3)(1-t^5)} \\ &= \frac{(1+t^2)(1-t^3)}{(1-t)(1-t^3)(1-t^5)} \\ &= \frac{1+t^2}{(1-t)(1-t^5)},\end{aligned}$$

which matches the given formula for $n=2$.

The general formula can be established by induction. The base case was just demonstrated. Now suppose that the formula holds for the sum of the first n terms. Adding the

$(n+1)^{\text{st}}$ term gives us the sum of the first $n+1$ terms. The resulting expression can be simplified as follows

$$\begin{aligned}
\text{formula} + (n+1)^{\text{st}} \text{ term} &= \frac{1+t^2+\cdots+t^{2n-2}}{(1-t)(1-t^{2n+1})} + \frac{t^{2n}}{(1-t^{2n+1})(1-t^{2n+3})} \\
&= \frac{(1+t^2+\cdots+t^{2n-2})(1-t^{2n+3}) + t^{2n}(1-t)}{(1-t)(1-t^{2n+1})(1-t^{2n+3})} \\
&= \frac{(1+t^2+\cdots+t^{2n-2}+t^{2n}) - (t^{2n+1}+t^{2n+3}+\cdots+t^{4n+1})}{(1-t)(1-t^{2n+1})(1-t^{2n+3})} \\
&= \frac{(1-t^{2n+1})(1+t^2+\cdots+t^{2n-2}+t^{2n})}{(1-t)(1-t^{2n+1})(1-t^{2n+3})} \\
&= \frac{(1+t^2+\cdots+t^{2n-2}+t^{2n})}{(1-t)(1-t^{2n+3})},
\end{aligned}$$

which is exactly what the formula predicts for the case $n+1$. Hence we are done by induction on n.

Students at Vestavia Hills high school discovered another neat method for proving this formula using a telescoping sum. Here is a hint: show that the n^{th} term can be written

$$\frac{1}{t^3-t}\left(\frac{1}{1-t^{2n+1}} - \frac{1}{1-t^{2n-1}}\right).$$

A–Part v B–Part v: Several schools demonstrated that the partial sums converged to the quotient shown. This wasn't intended to be part of the question, but the reason is not a great mystery. As $n \to \infty$ the numerator turns into the series $1+t^2+t^4+\cdots$, which equals $\frac{1}{1-t^2}$ since it is a convergent geometric series. The $(1-t)$ in the denominator stays put since it doesn't involve n, and the term $(1-t^{2n+1})$ tends to 1 since for $|t|<1$ we know that $t^{2n+1} \to 0$ as $n \to \infty$.

To find the coefficient of t^{13} we must expand the fraction into a product of infinite series:

$$\frac{1}{1-t} \cdot \frac{1}{1-t^2} = (1+t+t^2+\cdots)(1+t^2+t^4+\cdots).$$

At this point one can multiply the series together and discover directly that the coefficient of t^{13} is indeed 7. A clever way to conclude is to use the reasoning from part ii in reverse by arguing that this coefficient is the number of solutions (with nonnegative integers) to the equation $x+2y=13$. It is simple to list these solutions; they are (13,0), (11,1), ..., (1,6), for a total of 7 solutions. In general the coefficient of t^{2k-1} in the above product will be the number of solutions (in nonnegative integers) to the equation $x+2y=2k-1$. These are $(2k-1,0)$, $(2k-3,1)$, ..., $(1,k-1)$, for a total of k solutions, as surmised.

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